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ON THE POLARIZATION OF X-RADIATION

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In recent papers,[1] experiments on the general x-radiation produced by the impacts of high-speed electrons against the free atoms of mercury vapor have been described. In these experiments all the electrons had substantially the same velocity, as the difference of potential between the electrodes of the x-ray tube came from a high-tension storage battery. The absorption of the radiation in successive sheets of aluminum indicated that the radiation had the spectral distribution predicted by certain theories, in which distribution the intensity of the radiation has its maximum value at the short wave-length limit, given by the quantum relation, and then decreases beyond the limit toward longer wave-lengths inversely as the square of the wave-length.[2] It appeared, also, that the radiation had the same distribution whether it proceeded from the points of impact of the electrons in the direction of their motion or at right angles to this direction.

In a paper presented at the recent meeting of the American Physical Society in Washington the writer described experiments on the polarization of the radiation coming from the impacts of electrons against the mercury atoms. (See *Physical Review*, June, 1929, p. 1089.) The experiments indicated that the general (not the characteristic) x-radiation was largely but not completely polarized. This note describes the polarization experiments in greater detail and states some of the conclusions that may be drawn from them.

The x-ray tube employed to generate the radiation from the mercury vapor atoms differs from those previously employed only in a few details, chief among them being the fact that the mercury vapor is condensed by liquid air instead of by a water-cooling system. Figure 1 represents the glass x-ray tube. Electrons come from a Coolidge cathode, A, and pass through a circular opening into the anode, BB'. Metal rods, indicated by broken lines, connect the two metal parts of the anode, B and B'. Another rod connects the anode to a water-cooled cap, G, at the end of the tube,

this cap being metallically connected to earth. The cathode, A, is connected to a point on a manganine wire circuit having a total resistance of over 6,000,000 ohms (not shown in the figure). One end of this resistance is metallically joined to earth and, therefore, to the metal cap, G, and to the anode, BB'; the other end is connected to part of the high-tension storage battery. During the experiments to be described the voltage of this part of the battery amounted to approximately 18,000 volts, and the resistance of the manganine wire circuit was so regulated that a current of 2.90 milliamperes passed through it to the ground. The resistance of the manganine wire circuit between the point at which the cathode, A, was attached and the ground (i. e., the resistance of the circuit in parallel with the x-ray tube) amounted to 4,023,000 ohms and, therefore, the

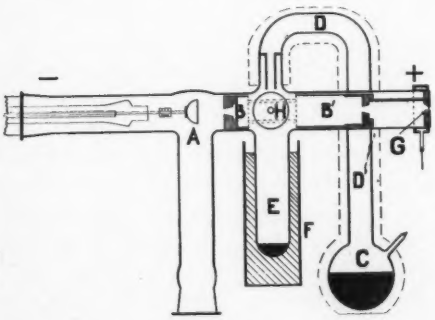


FIGURE 1

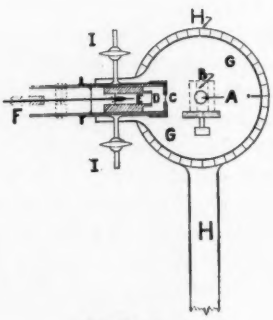


FIGURE 2

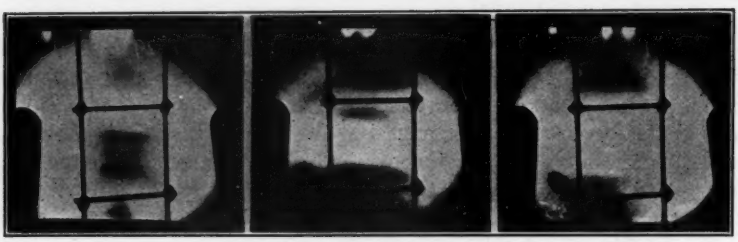
battery produced a constant difference of potential of 11,670 volts between the cathode and the anode. This voltage cannot produce any of the L series lines in the x-ray spectrum of mercury, for the generation of these L series lines requires at least 12,290 volts. The M series lines have such long wave-lengths (in the neighborhood of 6 Ångströms) and such large coefficients of absorption that they do not come through the windows of the x-ray tube with perceptible intensity. The spectrum of the radiation observed in the experiment, therefore, contained none of the x-ray lines and consisted of general or continuous spectrum radiation only. The reservoir, C, contains pure mercury, that is heated and vaporized by an electric furnace (not shown in the figure) upon which the reservoir rests. The mercury vapor passes up and over through the tube, D, and down through a constricted portion, as represented. The entire tube, D, is heated by means of an electric current flowing through a wire wound in asbestos around it. The coils of this wire lie considerably closer together near the constricted portion of the tube, which insures super-heating

of the mercury vapor and prevents the formation of drops of mercury in the central portion of the tube. From D the stream of mercury vapor passes down into the reservoir, E, where it is condensed by means of liquid air in the flask, F. With this arrangement of apparatus, a horizontal stream of high-speed electrons passes through a vertical stream of mercury atoms in the middle portion of the anode. The radiation coming from the impacts of the electrons against the mercury atoms in the direction of motion of the electron stream can be observed through the thin window, G. This window consists of a thin sheet of cellophane paper, which absorbs less than one per cent of the x-radiation. A magnet (not shown in the figure) produces a magnetic field in the region, B', of the anode,which deflects the electrons so that they do not strike the window, G, itself. The circles, H, represent a wide tube extending out toward the front of the diagram at right angles to the main tube. This tube carries a cellophane window similar to G through which the radiation coming from the mercury atoms in a direction perpendicular to the electron stream can be observed. The window, H, lies about 9 cm. from the axis of the electron beam. In the researches I am describing, the polarization of this radiation in the direction at right angles to the electron stream has been investigated. During the experiments, as high a vacuum as possible was maintained by means of a three-stage mercury vapor pump (not shown in the figure) on which the x-ray tube rested. Contact with this pump was made at the lower end of the tube, below A.

The polarization of the radiation has been examined by means of the scattered radiation coming from a block of carbon upon which the primary radiation impinged. Figure 2 represents the apparatus for detecting the scattered radiation projected from the block of carbon in directions at right angles to the primary x-ray beam. This horizontal primary beam of x-rays comes from the tube through a circular hole, A, in a vertical lead and brass disc, G, and strikes the block of carbon, B. The block is supported on a small metal disc, that can be turned about a vertical axis, which lies about 3 cm. in front of the circular hole, A. Part of the x-radiation scattered from the carbon block in a direction at right angles to the primary beam of x-rays passes through a cellophane window, C, into an air-tight chamber, D. It then passes through a very thin sheet of aluminum, D, into a metal box, E, containing a sharp metal point, usually of steel, but sometimes of platinum. The apparatus constitutes a "point counter" for counting photons that are absorbed in E. The writer first described this point-counter device in the *Comptes Rendus*, July 18, 1910. Since that date point counters have been greatly improved by Geiger and others. The point counter, the carbon block and its support are all attached to the lead disc, G, which can be rotated about an horizontal axis, parallel to the primary beam of x-rays. The disc is held in position by the

fixed ring and support, H. A circular scale on the ring enables the direction of the scattered rays coming from the carbon and entering the point counter to be determined. In the experiments to be described, the scattered radiation in the two horizontal directions, left and right and in the vertical directions up and down, was investigated. It will be remembered that the stream of electrons hitting the mercury atoms and producing the x-rays traveled in a horizontal direction, so that, if the point counter is in the position represented in the figure, or on the diametrically opposite side of the circle, it registers scattered radiation in directions parallel to the direction of motion of the electrons, and, if it is in either of the vertical positions, it registers x-radiation projected in a direction perpendicular to that of the electron stream.

By means of the glass stop-cocks, I, we are able to fill the point-counter



A B C

FIGURE 3

chamber with any desired gas. In these particular experiments we have used Argon. Argon increases the number of photons caught and registered in the point-counter chamber about eight times. In order to detect the current of electricity that flows to the point, E, due to the absorption of a photon in the chamber, the point is connected metallically through F and a flexible wire cable, F, to a three-stage amplifying system.[3] This amplification suffices to operate both a loud speaker and an electric relay, the electric relay operating an ink syphon recorder. Both audible and ink records have been taken simultaneously of the number of photons absorbed in the chamber E. In order to make the point counter work, it is necessary to apply to it a voltage of several hundred volts. The voltage required is less, when Argon fills the chamber, than when air fills it. Point counters, that require 2200 volts in air to work properly, work in Argon at about 1200 volts.

The photographs A, B and C of figure 3 were produced by the primary beam of x-rays that came through the window, H, figure 1, of the x-ray tube, the window thus constituting a kind of pin hole in a pin-hole camera.

The films were enveloped in black paper, which cut off the rays of ordinary light. They were placed at a distance of 4.5 cm. from the window, H, and immediately back of the hole, A, (Fig. 2) in the vertical disc. They were held in position by the cross wires, the shadows of which appear in the photographs. In the case of figure 3A, the mercury heater was not running and the dark part of the photographic impression (the light part of the reproduction) came from the impacts of a few stray electrons against the solid parts of the anode in the tube. Owing to the fact that these parts were solid and not gaseous, the impression produced by the x-rays are relatively very intense. In the photograph of figure 3A appears a faint but definite horizontal streak. The streak represents the x-radiation that came from the electron stream as it passed through the mercury vapor that existed in the tube at room temperature. Figure 3B represents the radiation produced under the same conditions, except that here the mercury heater had been running for several hours and the track of the electrons through the mercury vapor is very strongly marked. This photograph illustrates the way in which one can assure one's self that the radiation passing through the circular hole, A, in figure 2, really comes from the impacts of the electrons against mercury vapor atoms and not against solid matter inside of the tube. The hole, A, lay directly behind the center of the square marked out in the photograph by the shadows of the crosswires. The diameter of the hole, A, was less than one-half of the length of a side of the square and, therefore, the hole lay entirely in the region of the radiation coming from the mercury vapor itself, and hence only this mercury vapor radiation struck the carbon block producing the observed scattered radiation. Figure 3C represents a photograph taken some hours later, at which time the pressure of the mercury vapor in the tube had risen slightly.

In making a series of observations, it was customary to start the mercury heating circuit early in the morning and let it run for about five hours before beginning to take the readings. At that time, the thermal conditions, etc., had reached an approximately constant state. The electrical circuits of the high-tension battery were then closed and the Coolidge cathode put into operation and heated so as to give a current through the tube of a few milliamperes. The manganine wire potentiometer resistance was adjusted so as to give a current of 2.90 milliamperes through the portion of it parallel to the x-ray tube. The point counter was then placed vertically over the carbon block so as to record the effects of the scattered radiation in a direction perpendicular to the motion of the electrons. The number of photons absorbed in and recorded by the chamber during five minutes was then counted. Usually the counting took place not only orally by means of the loud speaker, but also by means of the ink syphon, recording on paper tape. The point counter was then moved to a hori-

zontal position and the absorbed photons counted for another five minutes. The same procedure was then continued with the point counter vertically below the carbon block and, again, with the point counter in the other horizontal position. Thus records were obtained of the numbers of scattered photons absorbed in the chamber that came from the carbon in four directions, two vertically upward and downward (i.e., perpendicular to the direction of motion of the electrons in the tube) and two in horizontal directions to the right and left (in directions parallel to the stream of electrons). The same process was then repeated five or six times, proceeding each time all around the circle.

A somewhat detailed description of one complete set of readings (experiment No. 33, on May 24th), taken with the loud speaker, will now be given. Column 1, of table 1, contains the time at which the counting

TABLE 1

EXPERIMENT NO. 33, MAY 24TH

TIME OF BEGINNING A SET OF FOUR COUNTS	NUMBER OF COUNTS BY LOUD SPEAKER IN FIVE MINUTES AT ANGLES INDICATED			
	0°	90°	180°	270°
1:40 P.M.	8	6	4	1
2:07 P.M.	115	41	109	36
2:32 P.M.	117	52	132	39
2:55 P.M.	95	43	92	33
3:18 P.M.	112	46	133	41
3:42 P.M.	111	43	141	52
Averages corrected for leaks	102	39	117	39

Degree of Polarization 0.474

began. Column 2 contains the number of photons recorded by the point counter during five minutes, when it lay vertically above the secondary radiator. Columns 3, 4 and 5 contain, respectively, similar counts of photons projected (*a*) in the direction of motion of the electrons, (*b*) vertically downward and (*c*) in the direction opposite to that of the motion of the electrons.

In this case, the heating current, started at 10:35 A.M., amounted to 1.05 amperes during the entire experiment. 3.55 milliamperes passed through the x-ray tube and 2.9 milliamperes passed through part of the potentiometer circuit in parallel with the tube. As the resistance of this parallel circuit amounted to 4,023,000 ohms, the voltage across the tube remained constant at 11,670 volts. This, as stated above, cannot produce any of the L series lines of the x-ray spectrum of mercury, for the generation of these L series lines requires at least 12,290 volts according to the well-known laws of x-ray production and, therefore, the x-rays contained general radiation only. The numbers in the first line represent the leak of the apparatus with the point counter in the four positions, respectively. They

were obtained with a sheet of lead covering the opening A (Fig. 2), cutting off the primary x-rays. The following lines contain the numbers counted when this sheet of lead was removed and the primary beam passed through, striking the secondary radiator, B. Subtracting the former numbers from the latter, we get, on the average, the number of photons absorbed by the counting chamber in its various positions due to the secondary rays coming from the secondary radiator. The counts

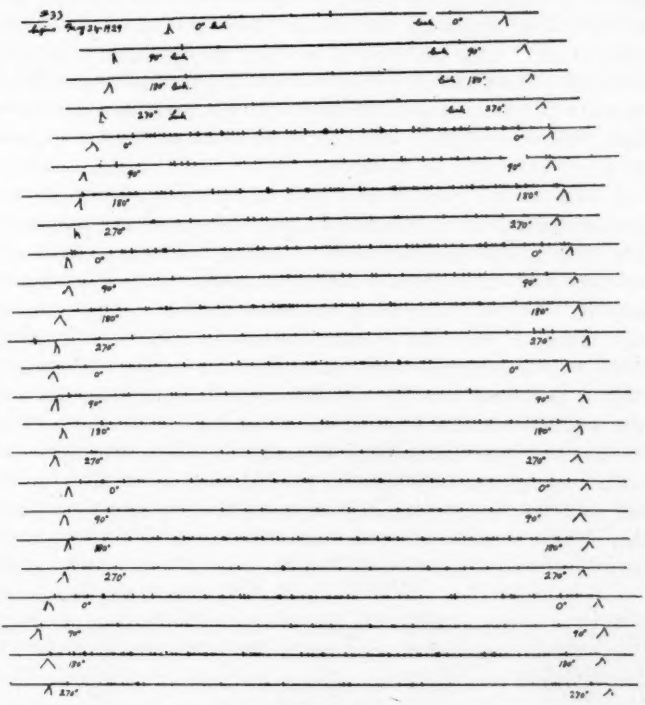


FIGURE 4

show conclusively that the numbers are nearly three times as great in the two directions up and down (perpendicular to the line of motion of electrons) as in the two directions forward and backward (parallel to the motion of electrons). In other words, the primary radiation is polarized to a considerable extent, since it produces more intense scattered radiation in directions perpendicular to the line of flight of the electrons than in directions parallel to it. The average numbers of photons absorbed in five minutes are $n_1 = 109.7$ in directions perpendicular to the electron stream and $n_2 = 39.1$ in directions parallel to the stream.

There are various ways in which the degree of polarization, p, of a beam of rays may be estimated and stated. For the present, we shall take the difference between the maximum number, n_1, of photons absorbed in a given time and the minimum number, n_2, divided by the sum of the two, namely, $p = \frac{n_1 - n_2}{n_1 + n_2}$. In this instance, the degree of polarization is $p = 0.474$.

Figure 4 contains a photograph of the record obtained with the ink syphon recording apparatus for exactly the same set of photon beams (Experiment 33). The ink syphon recorded the photons, while the observer counted the total number of the sound impulses coming from the loud speaker with a mechanical counter. All counts represent five-minute intervals. Those marked 0° and 180° record photons projected from the secondary radiator up and down, respectively (i.e., perpendicular to the electron stream); and those marked 90° and 270° record photons traveling in directions parallel to the electron stream. The first four records correspond to the leak of the instrument. Among the others, those marked 0° and 180° show a far greater number of photons absorbed than those marked 90° and 270°, indicating the same kind of polarization of the primary beam as the loud speaker indicated. The total numbers of absorbed photons recorded by the ink syphon were always less than those recorded at the same time by ear, showing that this particular syphon recorder was less sensitive than the loud speaker. The degree of polarization, however, was substantially the same in almost all cases. In this particular experiment, the degree of polarization determined by the ink syphon was 0.468, as compared with 0.474 determined by the loud speaker.

TABLE 2

SUMMARY OF ALL DATA

NUMBER OF EXPERIMENT	DATE	HEATING CURRENT AMPERES	TUBE CURRENT MILLIAMP.	LOUD SPEAKER		SYPHON RECORDER
				COUNTED IN 5 MINUTES	DEGREES OF POLARIZATION	DEGREES OF POLARIZATION
28	May 14	1.00 a.c.	3.4	94.4	0.482	0.382
30	May 20	0.93 a.c.	3.4	87.7	0.568	0.639
31	May 21	0.93 a.c.	3.4	75.1	0.507	0.494
33	May 24	1.05 d.c.	3.5	109.7	0.475	0.468
35	May 28	1.05 d.c.	3.5	154.0	0.478	0.495
36	May 31	1.05 d.c.	3.5	159.0	0.524	0.510
37	June 3	1.05 d.c.	3.3	183.5	0.472	0.351
38	June 5	0.75 d.c.	3.5	28.0	0.513	0.511
39	June 6	0.75 d.c.	3.5	35.3	0.587	0.581
40	June 7	0.75 d.c.	3.5	41.5	0.558	0.546
44	June 14	1.05 d.c.	6.2	205.0	0.428	...
46	June 18	0.80 d.c.	6.4	98.5	0.427	0.446

Table 2 contains a summary of all the data obtained after the preliminary runs made to adjust the apparatus, etc. Column 1 contains the number

of the experiment. Where a number is omitted, it means that an experiment was begun, but, for some reason (interruption of one or more of the currents, for instance), was broken off before completion. Column 2 contains the date; column 3, the current in amperes heating the mercury; column 4, the current in milliamperes through the tube; column 5, the average number of photons counted by the loud speaker in vertical directions during five minutes and column 6, the degree of polarization as defined above and as determined by the loud speaker. Column 7 contains the degree of polarization in the various experiments as determined by the ink syphon recorder.

It will be noticed that in all but two of the experiments the degrees of polarization determined by the loud speaker and the syphon recorder, respectively, agree with each other reasonably well, although, as stated above, the total number of photons counted by the loud speaker always exceeds the number counted by the syphon recorder. The average of the numbers representing the degree of polarization is 0.501 as determined by the loud speaker and 0.492 as determined by the syphon recorder. There seems to be no doubt that the radiation coming from the single atoms struck by electrons is very considerably polarized, but that it is not completely polarized.

The largest of the numbers representing the degree of polarization, as determined by the loud speaker, is 0.587, and the smallest is 0.427. The variation among the degree of polarization numbers, therefore, appears to be quite large. This variation is partly inherent in the method of counting, as the numbers of photons recorded in the point-counting chamber are in themselves variable. Some of the variation in the numbers representing the degree of polarization, however, is due to slight variations in the apparatus itself, such as slight differences in the amount of mercury vaporized per second, etc.

According to the Compton effect, part of the radiation scattered from the carbon block consists of modified and part of unmodified x-rays. The interesting question now arises, namely, whether both kinds of rays give the same indication as to the state of polarization of the primary beam. To obtain data bearing on this point, experiments were performed with chemical elements other than carbon as the scatterer; for the ratio of the intensity of the modified rays to that of the unmodified rays becomes much smaller for chemical elements of high atomic weight. However, in the experiments for chemical elements of high atomic weight, the number of photons registered in five minutes became so small that many more counts will have to be made before reliable conclusions can be drawn from them.

There is one source of error which would make the degree of polarization appear to be too small. When the electrons pass through the stream of mercury atoms, a certain number of them will be deflected and have their

velocities reduced in magnitude and these will strike other mercury atoms producing radiation corresponding to the slower velocities. Whether or not the error coming from this source is large enough to be perceptible can be determined, if we change the number of mercury atoms per cubic centimeter in the gas stream. If this number is twice as great, there will be twice as many electrons deflected and each deflected electron will have twice the chance of striking another mercury atom. The number of photons, therefore, produced by the electrons that have been slowed down will increase as the square of the density of the mercury atoms, whereas the number of photons produced by the impacts of full-speed electrons will increase as the first power of the density. We would expect, therefore, that the degree of polarization would depend upon the density of the mercury vapor. In some of the experiments recorded in table 2, the numbers of photons registered are several times as large as those registered in other experiments, due to the fact that the current heating the mercury was larger in some experiments than in others. If we compare experiments 33, 35, 36, 37 and 44 with experiments 38, 39, 40 and 46, we see that the numbers of photons recorded per minute in the first set of experiments average more than four times as large as in the second set. Hence, the density of the mercury vapor, on the average, must have been about four times as great in the first set as in the second. On calculating the degrees of polarization in the two cases, however, we find that the average degree of polarization for the denser experiments is 0.486, whereas, for the less dense experiments, the degree of polarization is 0.514. These figures differ from each other by less than six per cent, and it is doubtful whether the errors of experiment would not account for the difference, especially when we take into consideration the fact that the observer may miss some of the photons when they follow each other in rapid succession. We can, therefore, only conclude that if the change in density produces a change in the calculated degree of polarization at all, the change cannot be very great.

The number of counts made in the various experiments recorded in table 2 were not always the same. Hence, the various degrees of polarization recorded must carry with them different weights, when we calculate the average degree of polarization given by the data. A weighted mean value for the degree of polarization may be obtained by taking the total number of photons recorded in all experiments for each of the several positions of the point counter. These numbers are:

0°	90°	180°	270°
5351	1815	5696	1900

The total numbers give 0.497 as the weighted mean value of the degree of polarization, assuming it to be sensibly the same for all of the experiments under the various conditions of tube current, density of mercury vapor, etc.

The degree of polarization adopted in this note has been defined by

means of the numbers of photons registered by the point counter in different positions and not by means of the amount of energy radiated in different directions. If the radiations had been homogeneous, this would have made no difference. As mentioned above we have found that the radiation from the mercury atoms is not homogeneous. Therefore, the degree of polarization determined by the ratios of the numbers of photons registered is a kind of average degree of polarization for x-rays of different wave-lengths or for photons of different energy contents. We would expect the radiation from the mercury atoms to have the characteristics found by Ross[4] in his interesting experiments on the x-rays coming from a solid target. We would expect, namely, that the rays from the mercury atoms would become more and more completely polarized as we approach their short wave-length limit. In terms of the photon theory, this characteristic may be expressed by saying that the photons of greatest energy content would have the vectors in them (whatever the vectors are) all pointing in the same direction, whereas, for those photons in which the entire energy of the electrons are not taken over by the photons, the vectors may point in somewhat different directions.

According to the classical and other theories of the production of x-radiation, the radiation should be polarized in the plane containing the direction of motion of the electrons by which it was produced. In this respect the experiments on mercury vapor radiation substantiate the conclusions of the theory in part only. Radiation from the mercury vapor appears to be polarized in the proper direction, but only partially polarized. In a recent note to the Academy[4] Sommerfeld has generalized his interesting theory of the production of x-rays so as to include the possibility of their being only partially polarized and also the possibility of the projection of radiation in the direction of motion of the electrons.

When we regard the polarization of radiation from the point of view of the photon theory, many interesting problems present themselves—problems connected with the question of what we really mean by polarization and how the degree of polarization for non-homogeneous rays should be defined. If the degree of polarization is determined by the amounts of secondary radiation radiated in different directions, it may be that we shall have to regard the degree of polarization as a kind of probability problem.

It gives me great pleasure to thank my assistant, Mr. Lanza, for the care that he has shown in setting up various of the above described experiments and for the patience with which he has made the very tedious counts.

[1] Duane, William, these PROCEEDINGS, **13**, 662(1927); **14**, 450(1928).

[2] Webster, D. L., *Phys. Rev.*, March, 1923, p. 325; Kramers, *Phil. Mag.*, November, 1923, p. 836.

[3] Designed by Mr. Hewlett of the General Electric Co.

[4] *J. Opt. Soc.*, **16**, 375, June, 1928.

[5] PROC. NAT. ACAD. SCI., **15**, 393 (May, 1929).

ON MOSAIC CRYSTALS

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Communicated October 14, 1929

A. *Statement of the Problem.*—Very little progress has been made during the last decade toward a theoretical understanding of the physical properties of matter in the solid state. The reasons for this stagnation are very clearly presented in a recent paper by A. Smekal.[1] According to Smekal, the properties of solids can be arranged in two large groups as follows.

Properties of the first type are "structure insensitive." For a given substance, they are practically the same (no change in order of magnitude) whether we deal with single crystals or polycrystalline agglomerates. Examples are the density, the energy content, the thermal coefficient of expansion, the lattice structure as determined by x-rays, etc.

Properties of the second type are "structure sensitive," since they may be altered considerably by plastic deformation, by changing the size of the grain or by introducing small amounts of contaminations. They include the elastic limit, the breaking strength, the thermal and the electrical conductivity, etc.

Now it is well known that the theory of ideal lattices as developed by Laue, Bragg, Kossel, Born and others satisfactorily takes account of structure insensitive properties. Unfortunately, this theory is utterly helpless in regard to the structure sensitive properties. For instance, the values which have been deduced theoretically for the mechanical and the electrical strength are about 100 to 1000 times too large. But still more serious is the fact that the general phenomenon of plasticity of single crystals cannot be interpreted in any way on the scheme of ideal lattices. Such a lattice would indeed be perfectly elastic for any load not causing actual rupture.

Following A. Smekal, we may say that the central problem of the physics of crystals is to find a rational answer to the question, "How can we understand the coexistence of properties of the two groups mentioned above, making use only of the fundamental properties of the molecular building stones?"

In the following, I am sketching the main outlines of a theory which I think will provide the solution of the problem stated above. The most important feature will be, that ideal crystals, though dynamically stable, are thermodynamically unstable. *The thermodynamically stable forms are not completely characterized by the lattice structure as it is deduced from the x-ray structure analysis. On top of this primary structure, there is*

superposed a secondary structure, which shows perfect regularity also. In general it has to be interpreted as a slight periodic variation in density. The elementary spacing of this secondary structure will for most of the crystals be found in the region between 100 and 10,000 Å.

The solution proposed here for our problem is that all the properties of the first type are determined by the primary lattice, whereas the structure sensitive properties are mainly related to the secondary structure.

B. *Stability of Lattices—Mosaic Crystals.*—Let us define at first an ideal crystal. Such a crystal does not exist, but it will be a convenient starting point for our considerations. An ideal crystal satisfies the following two conditions.

1. Geometrically, it is described completely by one of the mathematically possible crystallographic groups. The elementary space of the lattice corresponds to that which is obtained by the x-ray structure analysis.

2. The total energy E of the crystal shall be a minimum against all infinitesimal variations of the lattice which transform this into a configuration satisfying again condition (1).

It must be mentioned that the possible variations of the type (2) do not represent independent conditions. Consider, for instance, a regular crystal which is built up by particles exerting forces on each other which are functions of the distance only. If for such a crystal E is a minimum, against a uniform compression it will be a minimum against every other variation of the type (1), shearing for instance.

Two important questions now arise. The first is whether the ideal crystal is really dynamically stable. In other words, is the energy a minimum relative to all small variations of the lattice, regardless of whether they are of the type (2) or not? In the second place, we want to know if the ideal crystal is also thermodynamically stable. The answer to this question will give us the sought-for information about the constitution of actual crystals. On it we concentrate our main interest. The first question also deserves much attention for reasons which will be mentioned later.

From the fundamental laws of thermodynamics, it follows that the thermodynamically stable configuration of an assembly of particles is characterized by the fact that its free energy is an absolute minimum. If we disregard the thermal agitation for the moment, the condition is that the total energy is an absolute minimum. Our results then will be correct in the neighborhood of the absolute zero point.

It must be understood that we are not interested for the present as to which different types of modifications of crystals may be obtained from given atoms, diamond and graphite, for instance, if we deal with carbon atoms. We are rather looking for finite, although not too large variations, which, when exerted on an ideal lattice will decrease its energy, the final

configuration, however, not being represented exactly by any crystallographic group, but only approximately so. I have pointed out one such variation in an earlier paper[2]) a variation which is equivalent to the formation of cracks. I have tried to extend the considerations given there, and I have found that the opening of cracks is not really the most fundamental effect in relation to our problem—I rather think, now, that it is secondarily related to a simpler phenomenon, namely, the secondary structure of crystals which was mentioned above.

I can indicate here only the general outlines of the reasoning which led me to the above conclusions. A complete quantitative discussion will be published in another place.

From the general phenomenon that a lateral contraction results when a crystal is extended, we conclude that a single lattice plane, if it were not sandwiched between other lattice planes would be characterized in the equilibrium state by a smaller spacing than the three dimensional lattice. Let us discuss the simplest case of a cubical crystal whose lattice constant is d_0. A single plane in equilibrium, then, is characterized by a spacing $d < d_0$. For NaCl, it can be deduced theoretically that $d = 0.94\, d_0$. In order to have an isolated plane with the spacing d_0, a certain uniform tension has to be applied, which may be, say, X dynes/cm. This suggests choosing as a variation of the ideal lattice a relative linear contraction δ of just one plane, say, II. If we consider only the forces exerted by particles belonging to this plane, the work done would be

$$dW_1 = -2SX\delta \qquad (S = \text{surface}). \tag{1}$$

Considering now the forces acting between II and the rest of the planes, it is evident that they will resist a contraction of II. If we calculate the work which we have to do to produce a contraction, we obtain in the first approximation

$$dW_2 = \kappa\delta^2 \tag{2}$$

where $k > o$. It can be seen very easily that there will be no term proportional to δ. From this we conclude that the total change of energy

$$dW = dW_1 + dW_2 < o$$

for a very small δ. It is clear that δ must be very small indeed; otherwise, such atoms of II which are very far apart would change their mutual distance by amounts larger than d_0, in which case, the relation (2) could not be written down directly. The argument can, however, easily be made rigorous by considering the following finite change of configuration. α) Contract II so that its spacing is changed from d_0 to d. β) Fill the gaps which have been opened by α) between the two parts of the crystal I and II, which are separated by II. γ) Rearrange the relative position

of I and II in a certain way. Then, if the total energy change for α), β), γ) is computed one finds, in general, that it is negative. Thus, the contraction of one definite plane of the crystal will lead to a thermodynamically favored state. Now this process can be repeated for a plane Π_1 which is far enough away from II that the effect of the contraction of II is not noticeable; then for a plane Π_2 and so on. The greatest gain

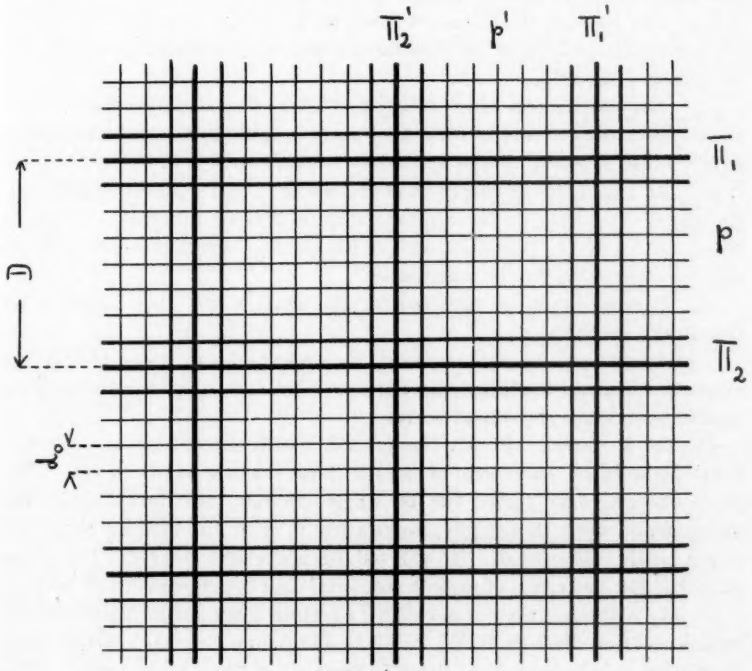

FIGURE 1

in energy will be obtained if, for the distance D between the planes, the smallest possible one is chosen. This means that the thermodynamically stable state will be characterized by constant spacings D. For our regular crystal, the argument can be extended to all three directions which leads to the conception of a secondary structure of the crystal, defined by a periodic change in density with the period D. A plane section is schematically shown in figure 1. The width of the lines represents line density. Our argument is based on the existence of a lateral contraction on extension. A secondary structure as described in the above should therefore be found in the great majority of crystals. As to the numerical value of D, the theory leaves open an approximate range from $D = 100$ Å to $D =$

10,000 Å, the tendency being for heteropolar crystals to favor the lower end of the scale, whereas homopolar substances will have large spacings. In regard to the space groups characterizing the primary and the secondary structure, it must be remarked that they are not necessarily the same, as will be elaborated in another place.

C. *On Optical Verifications of the Theory.*—It should be possible to test our theory by optical means in four ways.

(1) The pattern Π_i, Π_i' can be used as a surface grating. If D is in the visible range, it must be possible to observe the crossed grating in the microscope, directly.

(2) The phenomena which Bragg and Laue discovered for hard x-rays should have their analogue for longer wave-lengths (soft x-rays to visible light) if use is made of the secondary crystal structure.

(3) As the chemical energy of atoms along II and p planes is different, the speed of etching or of evaporation must also be different. In many cases, the resulting "fine structure" of the etching figures on single crystals will be observable in the microscope.

(4) Ghosts should be expected for the usual x-ray lines obtained by the Bragg method.

A systematic search was immediately undertaken in order to test the above predictions made by the theory. The following very interesting results have been arrived at so far.

a. Dr. A. Goetz, Mr. M. Hasler and myself discovered on a perfect (111) plane of Bi, three sets of parallel lines making angles of 60°. Dr. A. Goetz who had grown the Bi single crystals then investigated the phenomenon very thoroughly, correlating it with the etching effect (3) which he also discovered. He will publish his results shortly. I mention only that the pattern is of perfect regularity and corresponds to $D = 1.2\mu$.

b. My attention then was called to certain phenomena discovered by Stokes and described by R. W. Wood.[3] Potassium chlorate crystals often show what might be called "Bragg reflection" in the visible region. The crystals are perfectly transparent. If white light falls on the crystal at a definite angle, the reflected light is of a spectral color which depends on the angle of incidence. This and several similar phenomena I am inclined to interpret as Bragg reflection due to the secondary structure of the crystal.

c. Dr. Goetz found that single crystals of zinc after being etched produce a spectrum on illumination with white light. The effect can be observed in three positions due to the hexagonal structure of the crystal. A more complete investigation is under way.

d. Finally, I was able to identify effect (3) for Cu. Etching pictures of Cu single crystals have been published by the metallographic Laboratory of the Brown Boveri Co.[4] Equidistant steps in the etching grooves can very clearly be distinguished. The spacing is approximately $D = 2\mu$.

D. *Mechanical Properties of Single Crystals.*—The great ease with which slipping can be produced in many crystals has been a puzzle for a good many years. For rocksalt, for instance, the theory of ideal heteropolar crystals predicts a minimum load of about 10,000 kg./cm.^{2} which would cause slipping. The observed critical load, however, is only about 20 kg./cm.^{2} On our theory, the solution of the paradox is obtained at once. The planes Π_i, Π_i' which might be termed as being "out of phase" facilitate the process of slipping immensely. To quantitatively determine the critical shearing strength along a II plane for NaCl presents no difficulties, and agreement with the observations is obtained. It might be mentioned here that the formation of cracks which I have treated in an earlier paper[2] has probably to be considered as a secondary phenomenon, occurring preferentially along II planes.

Slipping usually results in a permanent deformation as the elementary blocks all along the slip planes will be turned irregularly. This provides immediately an explanation for the so far obscure plasticity of single crystals.

One very important feature of the theory is that it accounts readily for the hysteresis properties of single crystals. Our secondary structure Π_i, Π_i' will indeed change its dimensions with the temperature T. D will, in general, decrease if T is increased. But, actually, the transition from one equilibrium pattern to another will only take place if $T > T_i$ (T_i = ignition temperature). This follows from the fact that all the different patterns represent dynamically stable configurations. A certain thermal energy is, therefore, needed to cause the transition to the thermo-dynamically stable position. Dr. H. M. Evjen of this Institute has quantitatively discussed those questions related to the velocity of reaction for the case of rocksalt. His results will be published in the *Physical Review*. Schematically, we can represent the dependency of a definite structure sensitive property P on T as shown in diagram 2. P is a univalued function of T for $T > T_i$. In the region $T < T_i$, P may assume any value from a certain P_0 to P_i. The rectangular region, therefore, might be called the field of hysteresis phenomena. For NaCl, for instance, observations show that crystals obtained from the melt and crystals grown out of solution show different properties when investigated at room temperature, as should be expected from the theory. The observations also check the conclusion that D decreases with increasing T.

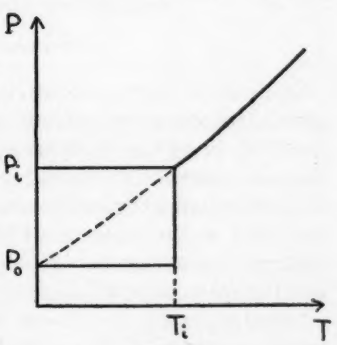

FIGURE 2

E. *Chemical and Electrical Phenomena.*—I have mentioned already that the average chemical energies along the p- and the II planes are different. This has an important bearing on chemical surface reactions, on electrode potentials, etc.

As A. Smekal has shown, the electrical conductivity of insulators is mainly determined by the imperfections of the crystal structure. The occurrence of imperfections, on the other hand, is essentially related to the secondary structure of the crystals which shows that the proposed theory promises a new view on a great number of interesting phenomena. A detailed account of the theory of mosaic crystals will be published in another place. The purpose of this paper is to outline the main features of the problem and its tentative solution.

[1] A. Smekal, *Zeits. Phys.*, **55**, 289, 1929.
[2] F. Zwicky, *Proc. Nat. Acad. Sci.*, **15**, 253, 1929.
[3] R. W. Wood, *Phys. Opt.*, 160, 1911.
[4] *The Brown Boveri Review*, January, 1929.

ON THE FOUNDATIONS OF RELATIVISTIC COSMOLOGY

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The general theory of relativity attributes the particular metrical properties of the space-time universe, considered as a 4-dimensional Riemannian manifold, directly to the distribution of matter within it, and has naturally led to speculations concerning the structure of the universe as a whole,in which the local irregularities caused by the agglomeration of matter into stars and stellar systems are disregarded. Chief among the resulting relativistic cosmologies are those based on the cylindrical world of Einstein[1] and the spherical world of de Sitter;[2] the line elements on which these interpretations are based have not, however, been derived from the intrinsic properties of homogeneity and isotropy attributable *a priori* to such an idealized universe, but rather are presented as defining manifolds which do possess the desired uniformity. It is the purpose of the present note to formulate explicitly an assumption embodying the uniformity demanded by such a cosmology and to deduce all line elements satisfying it.[3] We shall find that the only possible *stationary* cosmologies—i.e., the intrinsic properties of which are independent of time—are in fact those of Einstein and de Sitter, and that they arise from particular cases of a class of solutions whose general member defines a non-stationary cosmology.

We first introduce coördinates $x\alpha$ (Greek indices = 0, 1, 2, 3; we write $x^0 = t$) in which the line element assumes the form

$$ds^2 = dt^2 + g_{ij}\,dx^i\,dx^j \text{ (Latin indices = 1, 2, 3)}, \qquad (1)$$

where the $g_{ij}(t, x^1, x^2, x^3)$ define a negative definite form, in such a way that the matter in the universe has *on the whole* the time-like geodesics x^i = const. as world lines.[4] The coördinate t can then be interpreted as a mean time which serves to define proper time and simultaneity for the entire universe; in any portion t will be a mean proper time for all matter contained therein and the 3-space t = const. will be interpreted as the 3-dimensional physical space at time t.

Having described the actual universe in terms of these coördinates and having obtained thereby a natural and significant separation of space-time into space and time, we are now in position to state our only assumption, the assumption which is to express the intrinsic uniformity of the universe when local irregularities are disregarded. We demand that to any stationary observer ("test body") in this idealized universe all (spacial) directions about him shall be fully equivalent in the sense that he shall be unable to distinguish between them by any intrinsic property of space-time, and he shall similarly be unable to detect any difference between his observations and those of any contemporary observer; in the absence of irregularities the world offers no landmarks which enable us to distinguish between simultaneous events or spacial directions. Our assumption may be formulated mathematically:

I. *Space-time shall be spacially homogeneous and isotropic in the sense that it shall admit a transformation which sends an arbitrary configuration in any of the 3-spaces t = const., consisting of (a) an arbitrary point, (b) an arbitrary (spacial) direction through the chosen point and (c) an arbitrary 2-flat of (spacial) directions through the chosen point and direction, into any other such configuration in the same 3-space in such a way that all intrinsic properties of space-time are left unaltered by the transformation. That is, any such configuration shall be fully equivalent to any other in the same 3-space in the sense that it shall be impossible to distinguish between them by any intrinsic property of space-time.*[5]

Manifolds satisfying this condition alone will be suitable for a cosmology representing the ideal background of the actual universe—provided, of course, that they satisfy Einstein's field equations for some suitable choice of the matter-energy tensor—but they are not of necessity stationary. If we wish to require in addition that their intrinsic properties be independent of the time t, we may amend the above assumption to state that *any configuration*, as there described, *in one of the 3-spaces t = const. is fully equivalent to any such in any 3-space of the family.* That the transformations involved send spacial directions into spacial directions implies, as is readily verified, that the 3-spaces t = const. are transformed among themselves; with this in mind we may alternatively—and shall in the

following—characterize stationary space-times as ones which satisfy in addition to I the assumption:

II. *There shall exist a transformation of space-time transforming the 3-spaces t = const. among themselves and sending an arbitrary member of the family into any other without affecting the intrinsic properties of space-time. That is, any two of the 3-spaces t = const. shall be fully equivalent in the sense that it shall be impossible to distinguish between them by any intrinsic property of space-time.*[6]

In order to determine all manifolds satisfying I or I and II we first express these assumptions in the language of the theory of continuous groups. We note that I is satisfied if, and only if, it be possible to find a transformation which sends an arbitrary configuration into some standard one, so the transformations contemplated in I must consequently constitute a 6-parameter transformation group—corresponding to the 3, 2, 1 parameters required to specify the elements (*a*), (*b*), (*c*), respectively, of the configuration—which is intransitive and has as minimum invariant varieties the 3-spaces t = const. That two configurations are equivalent under one of these transformations implies that they, together with the unique time axes at the points of the configurations, may be taken as defining the origin and direction of axes of two coördinate systems with reference to which the line element of space-time assumes identically the same form. *This 6-parameter transformation group is consequently a group G_6 of motions of space-time into itself,*[7] and assumption I may now be stated:

I′. *Space-time shall admit an intransitive group G_6 of motions which has as minimum invariant varieties the 3-spaces t = const.* Similarly, II may be stated:

II′. *In order that space-time be stationary it shall possess a group G_1 of motions in which ξ^0 is non-vanishing and depends at most on the time t,* where ξ^0 is the time component of the infinitesimal transformation of the group.

Now Fubini has shown that a 4-space which admits an intransitive group G_6 of motions has as minimum invariant varieties a family of geodesically parallel 3-spaces of constant curvature which are mapped conformally on each other by their orthogonal trajectories.[8] In virtue of assumption I our line element (1) must consequently assume the form

$$ds^2 = dt^2 - e^{2f} h_{ij}\, dx^i\, dx^j \tag{2}$$

where f is an arbitrary real function of the time t and the coefficients h_{ij} are functions of the spacial variables x^1, x^2, x^3 alone such that the differential form

$$ds^{*2} = h_{ij}\, dx^i\, dx^j \tag{3}$$

is positive definite and defines a 3-space of constant curvature, say κ.

In order that this line element (2), which has been derived from the

single assumption I of spacial uniformity, be suitable for relativistic cosmology it must satisfy Einstein's field equations

$$R_{\alpha\beta} - {}^1/_2 g_{\alpha\beta}(R - 2\lambda) = -8\pi T_{\alpha\beta}$$

for some appropriate choice of the matter energy tensor $T_{\alpha\beta}$. But we find on inserting in (4) the values of the Ricci tensor $R_{\alpha\beta}$ computed for (2) that

$$T_{\alpha\beta} = (\rho + p)\delta_0^\alpha \delta_0^\beta - g^{\alpha\beta} p, \tag{5}$$

where

$$\begin{aligned} 8\pi\rho(t) &= \lambda + 3f'^2 + 3\kappa e^{-2f} \\ 8\pi\rho(t) &= \lambda - 2f'' - 3f'^2 - \kappa e^{-2f}, \end{aligned} \tag{6}$$

and this allows us to consider (2) as defining an ideal universe containing a uniform distribution of matter of proper density ρ and proper pressure p at rest with respect to the spacial coördinates x^i.[9]

Before proceeding to a discussion of the general line element (2) we apply assumption II′ in order to determine which cases of it lead to stationary cosmologies. The components ξ^α of the motion G_1 must satisfy the equations of Killing[7]

$$\xi_{\alpha,\beta} + \xi_{\beta,\alpha} = 0, \tag{7}$$

where $\xi_{\alpha\beta}$ is the covariant derivative of ξ_α with respect to x^β; in the case contemplated in II′ they are

$$\frac{\partial \xi^0}{\partial t} = 0, \quad h_{ij} \frac{\partial \xi^i}{\partial t} = 0 \tag{8a}$$

$$2\xi^0 f' h_{ij} + \xi^h \frac{\partial h_{ij}}{\partial x^h} + h_{kj} \frac{\partial \xi^k}{\partial x^i} + h_{ik} \frac{\partial \xi^k}{\partial x^j} = 0. \tag{8b}$$

Consequently ξ^0, which can depend at most on t, is a constant, say unity, and the ξ^k are independent of t. Since $f'(t)$ is then the only function in (8b) which could depend on t it must be a constant, say $1/a$, and we may take $f = t/a$ without loss of generality. Furthermore, the conditions of integrability of (8b) are[10]

$$\kappa f' h_{ij} = 0, \text{ i.e. } \kappa f' = 0, \tag{9}$$

from which it follows that either $f' = 0$, in which case we may take $f = 0$ and (2) defines Einstein's cosmology, or $\kappa = 0$ and $f = t/a$, from which direct computation shows that (2) has constant curvature $-1/a^2$ and defines de Sitter's cosmology.[11] *The only stationary cosmologies satisfying our requirements are those of Einstein and de Sitter.*

We now waive assumption II and return to the cosmology defined by the general line element (2). This (conformal-Minkowskian) space-time is of class one, i.e., it can be immersed in a 5-flat;[12] we shall actually express it in this form, which will be of value in discussing the macroscopic

properties of the universe. We take $\kappa > 0$ and write $\kappa = 1/R^2$; our spacial coördinates can then be so chosen that (2) assumes the form

$$ds^2 = dt^2 - e^{2f(t)}\left(\frac{dr^2}{1 - r^2/R^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\right). \qquad (10)$$

On defining

$$z_0 = \int (1 + R^2 f'^2 e^{2f})^{1/2} dt \qquad \begin{matrix} z_1 = e^f r \sin\theta\cos\varphi \\ z_2 = e^f r \sin\theta \sin\varphi \end{matrix} \qquad (11)$$

$$z_4 = Re^f(1 - r^2/R^2)^{1/2} \quad z_3 = e^f r\cos\theta$$

(10) may be written

$$ds^2 = dz_0^2 - (dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2) \qquad (12)$$

and the universe is represented by the general hypersurface of revolution about the z_0 or time axis

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = R^2 e^{2f} \qquad (13)$$

where $f(t)$ is expressed as a function $f[t(z_0)]$ of z_0 by inverting $z_0(t)$ as defined in (11).

Einstein's universe, for which f is constant, is represented as a cylinder with axis along the z_0 or time direction, and de Sitter's (in the form given in footnote 11 for $\kappa = 0$) by a pseudo-sphere. Unfortunately the stationary form of de Sitter's universe is here represented as a pseudo-sphere lying, as do all manifolds (2) for which $\kappa = 0$, in the infinite regions of the variables, z_0, z_4. To obtain a representation of these in the finite domain we may take z_1, z_2, z_3 as before and define

$$z_0 = \frac{e^f}{2a}(a^2 + r^2 + \alpha(t)),\ z_4 = \frac{e^f}{2a}(a^2 - r^2 - \alpha(t)), \qquad (14)$$

where

$$\alpha(t) = e^{-f}\int \frac{e^{-f}}{f'}\, dt.$$

The universe is then represented by

$$-z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = -e^{2f}\alpha \qquad (15)$$

where the function of t on the right is expressed as a function of $z_0 + z_4$ by the inversion of $z_0 + z_4 = a\, e^{f(t)}$. The stationary form of de Sitter's universe is, as is known, represented as a pseudo-sphere. The discussion of the remaining case, that in which $\kappa < 0$, can be obtained from the above for $\kappa > 0$ by replacing R, z_4 by iR, iz_4.

Finally, we consider briefly the question of Doppler effect in light from distant objects. We have found that the material content of our idealized universe is at rest, so the Doppler effect will be obtained by computing

the difference in time of arrival at the observer between two flashes of light emitted at times t_0, $t_0 + \Delta t_0$ from a point source at $P_0(x_0^i)$ at rest with respect to the spacial coördinates x^i. The spacial projection of geodesics in the space-time with line element ds^2 (2) are geodesic in the 3-space with element ds^{*2} (3), so light which leaves P_0 at time t_0 arrives at the observer at time t defined implicitly by

$$\int_{t_0}^{t} e^{-f(t)}\, dt = \int_{P_0}^{0} ds^*, \tag{16}$$

where the integral on the right is the geodesic distance P_0 and the observer, measured in the 3-space (3)—and is $R \sin^{-1} r_0/R$ in the coördinates employed in (10). Light leaving P_0 in the interval t_0, $t_0 + \Delta t_0$ will consequently arrive at the observer in the interval t, $t + \Delta t$ where $\Delta t = \Delta t_0 e^{f(t)-f(t_0)}$; the resulting Doppler shift $\Delta\lambda/\lambda = \Delta t/\Delta t_0 - 1$ will be attributed to a velocity of recession

$$v = c \tanh[f(t) - f(t_0)], \tag{17}$$

where t is defined by (16) and c is the velocity of light. Our choice of coördinates in the actual universe has been such that the above considerations apply to the residual Doppler effect, after averaging to eliminate the effect due to accidental "proper" motions. Of the two stationary cosmologies, only that of de Sitter will show such a residual effect, as in Einstein's f constant.

To summarize briefly our conclusions, we have described the actual world in terms of coördinates which effect a natural separation of it into space and time and have determined its ideal background by a single assumption (I) which is but the concrete expression of a uniformity implied by the very concept of a system of relativistic cosmology; in requiring that this ideal background can be fitted into the actual universe in the way described we have expressed in another way the assumption made by other writers on the subject, above all by H. Weyl,[13] that the world lines of all matter in the universe form a coherent pencil of geodesics. We have shown that our idealized space-time can be represented as a hypersurface of revolution in a 5-dimensional flat space, and that the additional requirement (II) that it be stationary leads to the cylindrical world of Einstein and the spherical world of de Sitter as the only possibilities. The material content of this idealized background of the actual universe, being at rest with respect to the spacial coördinates, leads to a unique Doppler shift which will appear in the actual universe as a residual effect.

[1] A. Einstein, *Sitzungsber. Berl. Akad.*, **1917**, p. 142.

[2] W. de Sitter, *Monthly Notices R. A. S.*, **78**, 3 (1917).

[3] A. Friedman, *Z. Physik*, **10**, 377 (1922); **21**, 326 (1924) and, more recently, R. C. Tolman, these PROCEEDINGS, **15**, 297 (1929) have also attacked the problem of deriving the most general line element suitable for relativistic cosmology. But Fried-

man's reduction to a normal form by means of his "assumptions of the second class" is unsatisfactory and, it seems to the present author, is not possible on his assumptions alone. Tolman, on the other hand, has restricted the form of the line element *a priori* and without taking full advantage of the isotropy which he mentions, and cannot deal with non-stationary possibilities (cf. loc. cit., p. 304). Both introduce untenable assumptions on the matter-energy tensor (cf. footnote 9 below) and require that Einstein's field equations be satisfied instead of making full use of the intrinsic uniformity of such a space, as we do here. The non-stationary solution found by Friedman is contained in (2) below as the case in which the proper pressure p vanishes.

[4] D. Hilbert, *Math. Ann.*, **92**, 15 (1924). Cf. also L. P. Eisenhart, *Riemannian Geometry*, p. 57 (Princeton, 1926). Prof. Eisenhart has remarked in a conversation with the author that for the purposes of the present paper we need not be so specific in introducing a coördinate system; we need only require that the time coördinate x^0 be chosen in such a way that the difference of x^0 between any two world points on the (geodesic) world line of any of the particles of which the material content of the universe is composed measure the proper time interval between the two events in question—or rather that x^0 measure proper time along a *mean* world line in each portion of the universe in the sense of our treatment above. Although in this more elegant treatment the remainder of the coördinate net may then be filled in at will, assumption I will lead to exactly the same result as in the above, viz., the line element (2) below, and the form of the resulting matter-energy (5) will tell us that the matter in this idealized universe is at rest with respect to the spacial coördinates x^i.

[5] We could in fact impose merely the requirement of spacial isotropy, i.e., that any such configuration be transformable into any other *containing the same point* (element a), without altering the results, as in either case space-time admits a group G_6 of motions (cf. below). Thus here, analogously to Schur's theorem, spacial isotropy implies spacial homogeneity. Furthermore, as follows from the sequel, it is only necessary to require that I be true for a given 3-space, say $t = 0$.

[6] It is to be noted that this use of the word "stationary" is at variance with that of Friedman, who defines (loc. cit., p. 380) a stationary manifold to be one, the coefficients of whose line element are independent of t. If we wish to reserve the word "static" to describe this circumstance, it is evident that a static manifold is stationary but the converse is not necessarily true; this would also seem at variance with Tolman's use of the word "static" (cf. loc. cit., p. 304).

[7] For an account of the theory of groups of motions consult Eisenhart, loc. cit., Chap. VI, in particular pp. 233ff.

[8] G. Fubini, *Annali di Matematica*, (iii) **9**, 64 (1904); cf. L. Bianchi, *Teoria dei Gruppi Continuii*. p. 544 (Pisa, 1918).

[9] Cf. Tolman, loc. cit., p. 298. But it is to be noted that whereas in our coördinates the world lines x^i = const. are actually geodesics, so that rest is a possible state for the material content of the universe, this is not the case in Tolman's or Friedman's coördinates. The condition that this be permissible in coördinates in which the g_{oi} vanish is that g_{00} depend at most on t, and is not satisfied in their discussion of the de Sitter universe.

[10] Eisenhart, loc. cit., p. 237; or, more readily, from eq. (69.8), p. 232, on noting that (8b) defines an infinitesimal conformal transformation of the 3-space with line element ds^* defined by (3).

[11] For a discussion of the de Sitter universe in terms of these coördinates see H. P. Robertson, *Phil. Mag.*, **5**, Suppl., May, 1928, p. 385, and also a forthcoming paper by H. Weyl in the same journal. (I have since discovered that these coördinates have also been employed by G. Lemaitre, *Jour. of Math. and Phys.*, **4**, 188 (1925), and

wish to take this opportunity to correct the omission of reference to this work in my previous paper.) It is to be noted, however, that this is not the only form of (2) which describes the de Sitter universe, as the line element for which $e^{t'} = a\sqrt{\kappa}\cosh(t/a)$ is also of constant curvature $-1/a^2$; although both ρ and p are constant in this case it does not represent a stationary cosmology in which t is interpreted as time. The stationary form can be obtained from this form by replacing t by $t + a\log(2R/a)$ and allowing $R \longrightarrow \infty$.

[12] These results follow, after some calculation, from the general conditions; for these consult Eisenhart, loc. cit., pp. 92, 197–198.

[13] Cf. H. Weyl, *Phys. Zeitschr.*, **29**, 230 (1923), and papers cited in footnote 11 above.

PRODUCTION OF HYDROXYL BY THE WATER VAPOR DISCHARGE

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Communicated October 3, 1929

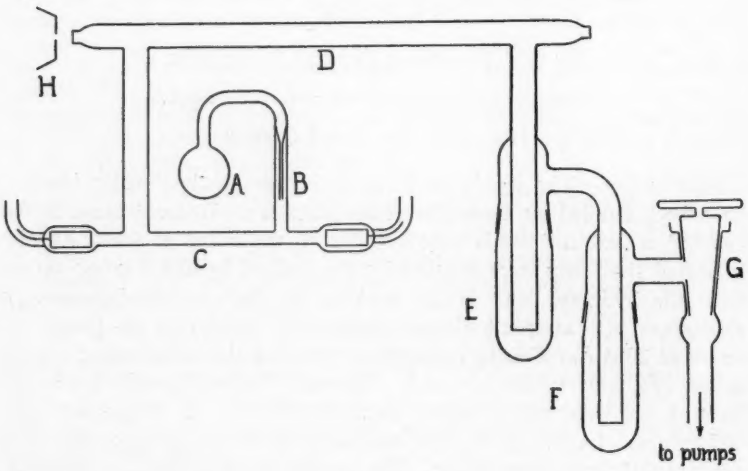
Introduction.—The bands at 3064 Å, known as the "water bands," have been studied by a number of investigators. Watson[1] came to the conclusion that the bands were not due to water but to OH. Mecke[2] indicated that they were emitted by the neutral hydroxyl group rather than the hydroxyl ion. While working on the chemiluminescence of substances with atomic hydrogen Bonhoeffer[3] noticed on his plates the bands at 3064 Å. This he explained by the fact that the hydrogen used in the discharge tube was moist. Recently Bonhoeffer and Reichardt[4] have shown that water vapor, heated to 1650°C., is dissociated into OH groups. Lavin and Stewart[5] have used the discharge-tube method to produce OH from water vapor. The present investigation is a continuation of the above work.

Apparatus.—The apparatus used is shown in the accompanying figure and is practically identical with that used by Urey and Lavin[6] in the preparation of atomic hydrogen from water vapor. *A* is a flask of distilled water, *B* a capillary. The discharge tube, *C*, was cooled in a bath of running water. *D* is the observation tube along which the photographs were taken; this tube was fitted with quartz windows. The traps, *E* and *F*, were provided with removable sections so that the condensed material could be taken from the system and analyzed. A Gaede all-steel, four stage, mercury vapor pump and several oil pumps completed the apparatus. A large Hilger quartz spectrograph was used.

Experimental.—The method of procedure in calibrating a capillary was to bring the water in *A* to a desired temperature, the system being evacuated all the while. CO_2 and ether mixture was kept around the second

trap F to protect the pumps from water vapor. When the apparatus had been well swept out liquid air was placed around trap E (which had been weighed previously) and the time noted. After a desired interval of time had elapsed the stopcock G was turned (to shut off the apparatus from the pumps), air let into the apparatus, and the trap removed and weighed. Between 0.2–0.3 gram of water was drawn through the apparatus in ten minutes.

When the discharge tube was on and the spectrograph, H, aligned with the observation tube, D, intense emission bands were observed. That these bands were not due to photographing the excited hydroxyl in the discharge tube itself may be demonstrated in a number of ways. Firstly,



only the hydroxyl bands were recorded on the plate; no lines due to atomic hydrogen (which are also present in the discharge tube) were observed. Secondly, when the discharge is allowed to take place in a static system, i.e., with the pumps cut off by stopcock G, no bands or lines are to be seen on the plates. And thirdly, if a cross-section of the observation tube is photographed at a distance from the entrance of the discharge tube the hydroxyl bands are obtained.

In order to determine the distance which these excited hydroxyls traveled down the observation tube a number of quartz windows were attached to the side of the tube at varying distances from the discharge tube inlet. The plates showed that the bands at 25 cm. were less intense than those obtained at 15 cm., no bands being recorded at a distance of 35 cm. in ten minutes. This seems to indicate that the hydroxyl groups are formed

in the discharge tube and are destroyed by some recombination reaction in the observation tube.

The water which was condensed in trap E (when the discharge was on) was shown to contain hydrogen peroxide. The plates and accompanying data of two typical runs are given below:

PLATE	TIME OF EXPOSURE (MIN.)	CC. $KMnO_4$ 0.1172 N.
1	10	0.60
	10	5.10
2	10	1.00
	10	0.80
	10	0.80
	10	0.75

Discussion.—From the data presented above it can be seen that the greater the intensity of the hydroxyl bands the greater the quantity of

PLATE 1

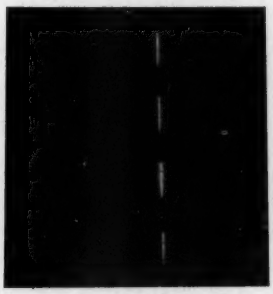


PLATE 2

hydrogen peroxide produced. Also, when there is little difference in the intensity of the bands the quantity of hydrogen peroxide obtained is about the same for the different runs. In the light of this it would seem that the reaction by which the hydrogen peroxide is formed must be the interaction of two hydroxyls or some other reaction in which one stage is the formation of excited hydroxyl groups.

Since the intensity of the hydroxyl bands seems to be an index to the quantity of hydrogen peroxide produced and since this can be determined by titration, a rough estimation of the hydroxyl concentration seems possible. Using the 5.10 cc. determination this would be 3.90 per cent of the water (0.2603 gram in ten minutes) passed through the apparatus. This is based on the assumption that the hydroxyl is formed in the discharge tube and that all of the hydrogen peroxide comes from this source. The experiments on the hydroxyl traveling down the tube, cited above, seem to be in accord with this assumption. Due to recombination with

atomic hydrogen and possible decomposition of the hydrogen peroxide already formed this is a minimum value.

The work is being continued.

The authors wish to acknowledge the many valuable suggestions of Dr. G. B. Kistiakowsky.

Summary.—The parallelism between the intensity of hydroxyl bands and the formation of hydrogen peroxide has been pointed out.

[1] Watson, *Astrophys. J.*, **60**, 145 (1924).
[2] Mecke, *Z. Physik*, **28**, 274 (1924).
[3] Bonhoeffer, *Z. Phys. Chem.*, **116**, 391 (1925).
[4] Bonhoeffer and Reichardt, *Z. Phys. Chem.*, **139A**, 75 (1928).
[5] Lavin and Stewart, *Nature*, **123**, 607 (1929).
[6] Urey and Lavin, soon to be published.

A SPECTROPHOTOMETRIC STUDY OF CLASS A STARS

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During the past two years a study has been in progress at Harvard of the absorption line contours and color temperatures of 119 stars of Class A. The results are published in detail in Harvard Circular 348, and a synopsis of the conclusions is given in the present note.

1. The total energy absorption of the *K* line of Ca^+ is found to be a good index of stellar temperature from Class *B*8 to *F*0. Considering the wide range of absolute magnitudes among *A* stars, and hence the probable wide range in pressure in the reversing layer from star to star, this seems at first glance to be almost too simple a finding, but the observational evidence points to it quite clearly, although perhaps a few exceptional cases exist.

A key to the explanation of this simple relation is contained in a recent note by Milne.[1] Considering stars of any given temperature higher than 8000° arranged in a sequence according to absolute luminosity, it is found from his theory that in the more luminous stars the decrease in the intensity of the *K* line which is to be expected on account of the decreasing pressure is almost precisely balanced by an increase in intensity on account of the greater depth of the reversing layer.

2. It is found that the Henry Draper classification is a surprisingly accurate index of temperature from *A*0 to *F*0. This is, of course, due to the fact that the criterion for these Henry Draper classes is largely the strength of the *K* line, which has been found to be a good temperature

index. The Mount Wilson classification,[2] on the other hand, is not primarily a temperature classification, the "*n*" classes being systematically cooler than the "*s*" classes, and the temperature range within any one class considerably greater than for the Henry Draper classes.

3. Among stars of any given temperature in the range from *B*8 to *A*2 it is found that the contour of the hydrogen lines depends on the luminosity of the star, the lines growing progressively wider for their depth as we pass toward the less luminous stars. This gives a fairly reliable criterion for absolute magnitude in stars from *B*8 to *A*2.

The widening is interpreted as a manifestation of the Stark effect, the less luminous stars having the higher pressures and hence larger values of q (the number of charged particles per unit volume). Milne has found, according to his theory, which neglects the effects of radiation pressure and of q, that at any given low temperature an increase in absolute luminosity is accompanied by an increase in the intensity of the hydrogen lines but that at temperatures around 10,000° and higher temperatures the increase should theoretically become negligible.

It seems improbable that q is negligible in the case of hydrogen line contours in *A* stars, in view of the evidence of Struve[4,5] and Elvey[6] that the Stark effect is detectable in the helium lines in *B* stars, and also in view of Unsöld's evidence[7] that the Stark effect accounts satisfactorily for the broad shallow character of the later members of the Balmer series in the solar spectrum. Hence, since the hydrogen lines become wider in the less luminous *A* stars, instead of remaining unchanged, as they would according to Milne's theory if q were negligible, it is inferred that the Stark effect is responsible for the widening of the Balmer lines in *A* stars.

4. The "peculiar" *A* stars in which the helium lines are prominent and the *K* line is as strong as in *A*0 stars are satisfactorily explained according to Milne's theory. For according to the contours of the hydrogen lines these are stars of relatively low atmospheric pressure, in whose spectra the helium lines should theoretically be strengthened and the intensity of the *K* line remain unchanged. The predicted effects are exactly as observed.

5. In the study of color temperature two unexpected phenomena were uncovered. There is a considerable apparent variation in color index with season, under Cambridge climatic conditions, stars appearing redder in the summer months; thus the color of an *A*0 star in summer is the same as the color of an *A*2 star in winter.

The *cA* stars that are abnormally red for their spectral class have a peculiar spectral energy distribution such that the "temperature" of a given star depends on the region of the spectrum investigated. For example, the "temperature" of μ Cephei, a *cA*2 star, as determined from certain spectral regions to the red of 4000*A*, is 1500° less than the "tem-

perature" as determined from spectral regions to the violet of 4000*A*. A similar phenomenon in *B* stars has already been referred to by Gerasimovič,[8] who speaks of the stars in question as being cool stars with an "ultra-violet appendage."

[1] Milne, *H. B.* 870, 1929.
[2] Adams and Joy, *Ap. J.*, **56**, 242, 1922.
[3] Milne, *Proc. Roy. Soc.*, (in press), Bakerian Lecture, 1929.
[4] Struve, *Ap. J.*, **62**, 198, 1925.
[5] Struve, *Ibid.*, **69**, 173, 1929.
[6] Elvey, *Ibid.*, **69**, 237, 1929.
[7] Unsöld, *Ann. d. Physik* (a paper on the pressure effect on the hydrogen lines in the sun, to appear shortly).
[8] Gerasimovič, *H. C.* 339, 1929.

GENETIC FACTORS STIMULATING MUTABILITY OF THE MINIATURE-GAMMA WING CHARACTER OF DROSOPHILA VIRILIS

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The mutable miniature gene of *Drosophila virilis* is in a labile condition reverting frequently to its wild type allelomorph.[1] These reversions occur at any stage in the development of the fly, in the germ cells as well as in the somatic cells. Recently several lines of mutable miniature were isolated, in each one of which the mutability of the miniature gene manifests itself at certain particular stages in the development of the organism. In the alpha line, the miniature gene is mutable both in the germ cells and in the somatic cells; in the gamma line, the mutability is limited to the somatic cells only; and in the beta line, the gene remains almost constant.

The experiments to be described in this paper involve only the miniature-gamma line. The somatic reversions occurring in the cells of the wings of the flies of this line produce wings which are mosaics of miniature and wild type tissues.

In an earlier paper[2] evidence was presented on the inheritance of two dominant genes which stimulate the mutability of mutable miniature. One of these genes, known as *M*, stimulates the mutability in the miniature-alpha line; that gene, however, does not influence the mutability of the miniature-gamma gene. The other gene, originally named *S*, but now known as *S*-1, stimulates the mutability of the miniature-gamma gene.

In the same paper evidence has been presented which indicates that *S*-1 is linked with the gene for "rounded-wing," a character located in the second chromosome.

In this paper data will be given on the inheritance of two additional genes which stimulate the mutability of the miniature-gamma gene. These two genes were discovered when the stock cultures available in the laboratory were tested for possible modifiers of mutability of the miniature-gamma gene.

THE *s*-2 GENE. *Origin.*—A recessive *s*-2 gene which stimulates the mutability of miniature-gamma was isolated from a stock culture of wild type flies which were direct descendants of flies collected in 1926 in New Orleans by Doctor C. I. Bliss. Except for the stimulating effect on the mutability of the miniature-gamma gene no other effect of the *s*-2 gene has been observed.

Inheritance.—In the heterozygous condition, the *s*-2 gene behaves as a complete recessive, and apparently has no stimulating effect on the miniature-gamma gene. In table 1 data are presented from a cross between

TABLE 1

F_1 OFFSPRING OF A CROSS *mt*-γ $\times$ *mt*-γ *s*-2/*s*-2

CULTURE NUMBER	MOSAIC	MINIATURE	TOTAL
11269	6	123	129
11270	5	112	117
11432	3	56	59
TOTAL	14	291	305

miniature-gamma females homozygous for the dominant wild allelomorphs of the *s*-2 gene and miniature-gamma males homozygous for the recessive *s*-2 gene. Out of a total of 305 flies heterozygous for *s*-2/+, 14 or 4.59 per cent were mosaics and all remaining ones were miniature. The line from which the miniature-gamma females were taken was propagated by brother-sister matings and observations on the number of mosaics were made during several generations. During the four generation previous to the experiment this line produced 4.84 per cent of mosaics in a total of 1942 individuals and the family from which the females were taken had 18 mosaics and 296 miniature flies, viz., it had 5.73 per cent of mosaics. Since miniature-gamma with *s*-2 in heterozygous condition produced 4.59 per cent of mosaics, it is evident that the heterozygous *s*-2 gene did not increase the number of mosaics and hence did not change the mutability of the miniature-gamma gene.

The F_2 data of the crosses with *s*-2 are given in table 2 from which it may be seen that the miniature and the mosaic flies segregated in a 3:1 ratio, the difference from expectation being only 13 ± 8.24.

Eleven cultures of miniature-gamma flies homozygous for the *s*-2 gene gave a total of 542 mosaic and 29 miniature flies indicating that the homo-

zygous *s*-2 gene stimulated the mutability of the miniature-gamma gene in a large proportion of the flies in which it was present.

Independence of s-2 from S-1 and S-3.—The fact that *s*-2 is a recessive and *S*-1 and *S*-3 are dominants is assumed at present as sufficiently strong

TABLE 2

F_2 DATA FROM A CROSS *mt*-γ × *mt*-γ *s*-2/*s*-2

CULTURE NUMBER	MOSAIC	MINIATURE	TOTAL
11577	21	52	73
11578	28	99	127
11579	39	113	152
11580	46	125	171
11581	49	102	151
11582	29	93	122
	212	584	799
CALCULATED 1:3	199	597	
DIFFERENCE	13 ± 8.24		

evidence for a preliminary assumption that *s*-2 is not an allelomorph of either *S*-1 or *S*-3. The final conclusion on this point, however, will be reached when the results of linkage tests, now in progress, become available.

THE *S*-3 GENE. *Origin.*—The dominant *S*-3 gene stimulating the mutability of miniature-gamma was isolated from a stock culture homozygous for sex-linked characters sepia, yellow, scute, cross-veinless and vermillion. As in the case of *S*-1 and *s*-2, *S*-3 does not have any noticeable morphological effect on the flies.

Inheritance.—As may be seen from F_1, F_2 and back-cross data presented in tables 3, 4 and 5, *S*-3 behaves as a completely dominant gene.

TABLE 3

F_1 DATA FROM A CROSS *mt*-γ × *S*-3/*S*-3

	FEMALES	MALES	
CULTURE NUMBER	+	MOSAIC	MINIATURE
10942	49	28	8
11605	32	27	..
	—	—	–
TOTAL	81	55	8

Independence of S-3 from S-1.—Since both *S*-1 and *S*-3 are dominant factors, the test of their linkage relations is the easiest way of determining whether they are genetically different or not. It has been shown in a

TABLE 4

F_2 DATA FROM CROSS *mt*-γ × *S*-3/*S*-3

CULTURE NUMBER	+[3]	MOSAIC	MINIATURE
11350	132	85	35
EXPECTED 3:1 ratio		90	30
DIFFERENCE		5 ± 3.2	

previous paper[2] that *S*-1 is linked with a dominant rounded-wing character. The data given in table 6 indicate that *S*-3 is independent of rounded, and therefore not an allelomorph of *S*-1.

TABLE 5

BACK-CROSS DATA FROM CROSSES *mt*-γ × *mt*-γ *S*-3/+

CULTURE NUMBER	MOSAIC	MINIATURE	TOTAL
11737	73	67	140
11738	39	42	81
11739	76	47	123
11649	80	133	213
11650	46	35	81
TOTAL	314	324	638
CALCULATED 1:1	319	319	
DIFFERENCE	5 ± 8.52		

TABLE 6

BACK-CROSS DATA *mt*-γ × *mt* *S*-3/+ *R*/+ INDICATING INDEPENDENT INHERITANCE BETWEEN *S*-3 AND *R*

CULTURE NUMBER	PARENTAL COMBINATIONS		RECOMBINATIONS	
	R +(MINIATURE)	+ *S*-3 (MOSAIC)	*R S*-3 (MOSAIC)	+ +(MINIATURE)
11737	32	33	40	35
11738	24	18	21	18
11739	21	34	42	26
TOTAL	77	85	103	79
GRAND TOTAL	162		182	
CALCULATED 1:1	172		172	
DIFFERENCE	10 ± 6.25			

DISCUSSION.—The major part of the laboratory material of *Drosophila virilis* traces its origin to three collections of wild type flies (New York City, 1913; Terre Haute, Indiana, 1919 and New Orleans, 1926). The flies from the first two collections were used extensively by Dr. C. W. Metz in genetic experiments and the stocks are at present thoroughly intercrossed.[4] The stock of the New Orleans collection was propagated by inbreeding. In 1927 several stocks of genetic material collected in Japan were obtained from Professor Taku Komai.

Of the three known factors, which stimulate the mutability of the miniature-gamma gene, two, *S*-1 and *S*-3, were found among the stocks which descended from the New York and Terre Haute collections, and one factor, *s*-2, was found among the descendants of the New Orleans collection. The stocks obtained from Japan did not carry any gene affecting the mutability of miniature-gamma. The modifiers of the mutability of miniature-gamma gene might have originated in the laboratory by mutation or might have been present in the flies collected in nature. In case

they were present in the collected material the frequency of them was fairly high, three having been found in the four collections.

SUMMARY.—Data are presented on the inheritance of two additional genes which stimulate the mutability of the mutable miniature-gamma gene. One of these genes (*s*-2) is a simple Mendelian recessive and the other (*S*-3) is a dominant factor.

[1] Demerec, M., these PROCEEDINGS, **13**, 1927 (249–253).

[2] Demerec, M., "Verh. V internat. Kongr. f. Vererbungswis.," vol. 1, 1928 (183–193).

[3] The wild class in this experiment comes from the wild type allelomorph of miniature brought into the cross by the male parent.

[4] Metz, C. W., Moses, M. S., and Mason, E. D., *Washington, Carnegie Institution Publ.*, No. 328, 1923.

THE OCCURRENCE OF A HERITABLE TWISTED NOSE IN THE HOUSE MOUSE, MUS MUSCULUS

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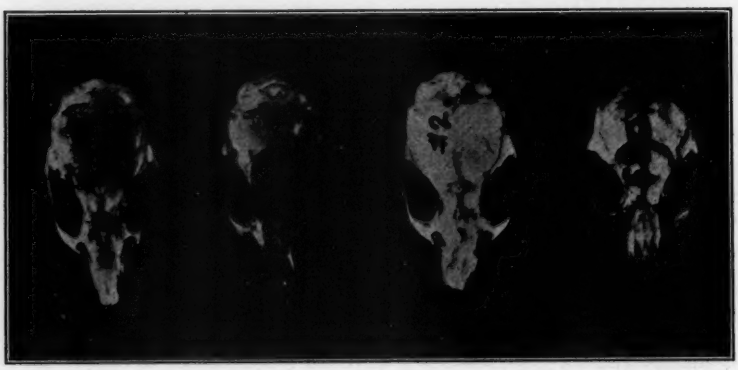
In a black-silver laboratory strain of mice a female (No. 663) was found, the left nasal bone of which was quite shortened. As the right nasal bone



1 2 3 4

FIGURE 1

grew to normal length, the adult nose was twisted to the left (Fig. 1, No. 2).

A normal nosed black-silver brother to No. 663 was mated to a wild house mouse (Fig. 1, No. 1) caught in Forest Hills. Among the seven

progeny there was found a male No. 2 (Fig. 1, No. 3) which bore distinctly the same abnormality except that the twist was to the right.

This male was mated to a normal nosed sister and among their offspring was found male No. 1085 (Fig. 1, No. 4) in which both nasal bones were shortened. The genetic relationships between these animals are shown in figure 2.

The data are insufficient to establish the exact mode of inheritance. The character is so rare that it is very improbable that the wild stock carried it as a recessive character. One possible explanation is that of a dominant unit character with normal overlaps.

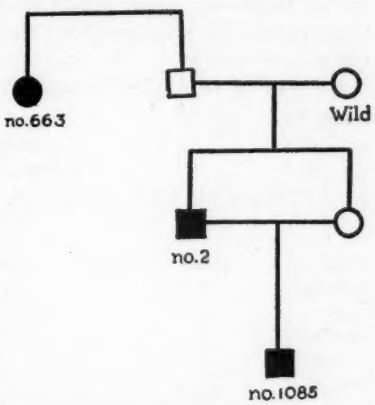


FIGURE 2

Pedigree Chart showing the transmission of a twisted nose through three generations in the House Mouse.

ON THE AMOUNT OF EXTERNAL MIRROR IMAGERY IN DOUBLE MONSTERS AND IDENTICAL TWINS

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Communicated October 9, 1929

Morrill,[1] who in 1919 reviewed the literature upon internal imagery in double monsters, mentioned that there were no data available upon external characters. Wilder[2] examined a case of conjoined twins (Margaret and Mary). In this case there was no asymmetry in the hands and therefore the nature of the imagery could not be determined. But the left sole of Mary was more nearly like the right sole of Margaret than like her own right sole and hence this may be considered a case of mirroring.

Mirror imaging has long been known in twins and has been well studied. With difficulty, one is able by the aid of a binocular to make out the friction patterns upon the hands and feet of foetuses of seven months or more gestation. The finger tips are usually much wrinkled in uninjected specimens, but by persistence one is able to pinch them flat long enough to record the gross form of the pattern. Head whorls are developed and ear shapes are distinct.

On account of the paucity of data available, it seems worth while to put on record our observations made on foetal monsters, comparing the frequency of mirroring here with that found in a sample series of identical twins.

A series of 14 pairs of identical twins was examined in Boston (see table 1).

TABLE 1

IDENTICAL TWINS

(Spaces left blank indicate lack of asymmetry in the character)

NAME	HANDS	TEETH	HAIR WHORL	EARS	NUMBER OF MIRRORED CHARACTERS DETECTED	NUMBER OF IDENTICAL CHARACTERS DETECTED
Hamilton	*I*		*I*		0	2
Segel	*I*		*I*		0	2
Lightbody	*M*		*I*		1	1
Spector			*I*		0	1
Holland			*I*		0	1
Levy	*I*		*M*		1	1
Morris		*I*	*I*	*M*	1	2
Donnelly			*I*		0	1
Dooling			*I*		0	1
Dooley	*I*	*I*	*M*	*M*	2	2
Cassidy			*I*		0	1
Stall			*I*		0	1
O'Leary			*I*		0	1
Marshall			*I*		0	1
Total					5	18
Total in %					22%	78%

The Lightbody twins show mirroring in the hands, but identity in the head whorl. The Levy twins show mirroring in the head whorl, but identity in the hands. The Morris twins show mirroring in the ear-lobe, but identity in hands and a crooked canine-tooth. The Dooley twins are mirrored in whorl and ears, but identical in teeth and hands. The Hamilton and Segel twins show identity in hands and hair whorls. The remaining eight pairs of twins show identity in hair whorls. Out of 23 asymmetrical characters 5 or 22% were mirrored. In this series, out of 14 pairs of identical twins four pairs showed some mirroring.

The twins exhibiting mirrored hand patterns were both right-handed, indicating that handedness depends upon cerebral dominance and not upon hand form.

A series of ten double monsters was studied. We have separated them into two groups: those joined on the side and those joined ventrally or by the head.

Table 2 shows the number of mirrored and identical characters detected in double monsters joined at one side. In the six double monsters

TABLE 2

DOUBLE MONSTERS JOINED AT SIDE

(Spaces left blank indicate lack of asymmetry in the character)

SPECIMEN	HANDS	FEET	HAIR WHORL	EARS	NUMBER OF MIRRORED CHARACTERS DETECTED	NUMBER OF IDENTICAL CHARACTERS DETECTED
Carnegie 1616	*M*	*M*	*M*		3	0
Carnegie 3328	*M*	*M*	*M*	*M*	4	0
Carnegie 2107	*M*	*M*	*I*		2	1
Harvard 5834	*I*				0	1
Harvard 10477	*M*				1	0
Warren Dr. Perley	*I*				0	1
Total					10	3
Total in %					77%	23%

joined upon the side we have found 10 or 77% mirrored and 3 or 23% which are identical. Four monsters out of six showed external mirroring.

TABLE 3

DOUBLE MONSTERS JOINED VENTRALLY

SPECIMEN	HANDS	FEET	HAIR WHORL	EARS	NUMBER OF MIRRORED CHARACTERS DETECTED	NUMBER OF IDENTICAL CHARACTERS DETECTED
Dr. Parker	*I*		*I*	*M*	1	2
Warren Dr. McCollem	*I*	*I*	*I*		0	3
Carnegie 3713	*I*	*I*	one head	two ears	0	2
Warren Dr. Torbert					0	0
Total					1	7
Total in %					13%	87%

The results for the ventrally joined monsters are tabulated in a similar fashion in table 3. Among three ventrally joined monsters showing asymmetries, one pair of characters is mirrored and 7 pairs are identical. One monster out of three showed mirroring.

Although granting the high probable error present through the meagreness of our data, nevertheless, there appears to be a distinct difference in the relative numbers of mirrored and identical characters correlated with the two modes of union. Among the laterally joined monsters, 10 out of 13 asymmetrical pairs of characters or 77% are mirrored. Among the ventrally joined monsters 1 out of 8 asymmetrical pairs of characters or

13% were mirrored. This may be contrasted with our findings for identical twins where 5 out of 23 asymmetrical characters or 22% were mirrored.

It is well established that gross internal mirrorings in double monsters are more frequent than in identical twins. It may be conceived that early and complete separation of twins leaves each individual independent to develop the asymmetries determined by their identical heredity. Where later fission or incomplete separation takes place the dual organism develops as a whole with mirror tendencies, some of which may be overcome by the normal body orienting tendency of the individual.

Since one individual of a pair of twins or a monster is normal with regard to the orientation of unpaired asymmetries such as the head whorl and viscera, etc., whereas the other may have any condition between complete reversal of all patterns and the normal orientation, we may think of the former individual as the pattern and the second as the mate influenced toward mirror imaging by its relative position, orientation and time of separation from the pattern individual.

In two of our monsters joined upon the side, the left individual was the pattern or normally oriented twin as indicated by the reversal of head whorl in the right twin. The other specimens gave no data upon this point, since both individuals had similarly oriented whorls.

Summary.—1. A series of 10 double monsters was examined for external asymmetries. Out of 6 monsters joined upon the side, mirrored characters were found in 4. Among these there were found 10 mirrored pairs of characters and 3 identical pairs.

2. Among 4 double monsters joined ventrally or by the head, 3 showed asymmetries of which 1 pair of characters was mirrored and 7 pairs were identical.

3. Among 14 pairs of identical twins all showing asymmetries, 4 pairs showed some mirroring. Among all the twins together, 5 characters or 22% were mirrored, while 18 or 78% were identical.

4. Some mirroring was found in 29% of the twins, 33% of the monsters joined ventrally and 66% of the monsters joined laterally.

5. Relative position, orientation and time of separation may be factors involved in determining mirroring both in double monsters and in identical twins.

I am indebted to the Warren Museum of the Harvard Medical School, Dr. G. L. Streeter of the Carnegie Institution, and Professor G. H. Parker for specimens employed in this study.

[1] Morrill, C. V., *Anat. Record*, June, 1919.

[2] Wilder, H. H., quoted from H. H. Newman, *The Biology of Twins*, Univ. of Chicago Press, Chicago, 1924.

MEIOSIS IN A TRIPLOID FRAGARIA

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In a cross between *Fragaria bracteata* and *Fragaria vesca rosea*, two closely related species having seven pairs of chromosomes, a tetraploid appeared (Ichijima).[1] This plant, when selfed, produced only tetraploids—seven plants in all. These 28-chromosome plants were crossed reciprocally with *F. vesca*, with *F. bracteata* used as female, and with two other 14-chromosome species used as male, namely, FPI64856, a native of China, and *F. collina*, a European type. All of these crosses resulted in families consisting of plants having the triploid number (21) or thereabouts.

Buds have been examined cytologically from all of the crosses except that involving *F. bracteata*. The chromosomes, instead of forming seven trisomes or seven disomes plus seven monosomes, usually arrange themselves into groups of ten disomes plus an unpaired chromosome. Not infrequently a secondary association occurs, however, which produces three groups of four, one group of three and three groups of two chromosomes. This secondary association is rather loose except in the case of the trisome, though occasionally first metaphase plates are found which appear to show counts of between seven and eleven chromosome formations.

At the second metaphase, where both plates can be counted, ten and eleven chromosomes are found most frequently. This is likewise true of counts made at second anaphase. When any exceptional behavior was noted, it appeared to be due to non-disjunction, a phenomenon found more frequently in the triploids than in the tetraploids. Thus one cannot escape the conclusion that there is complete pairing between non-homologous chromosomes. It is the logical deduction from the counts at diakinesis, and is supported by the fact that chromosomes of different sizes are paired. Moreover, this type of diaphase results in functional gametes, for the plants are partially fertile.

Only two references to this type of triploid meiosis have been discovered. Gates has reported complete pairing in two triploid Oenotheras, one in the cross *O. lata* × *O. gigas*[2] and the other in the cross *O. rubricalyx* × *O. gigas*.[3]

Pairing of homologous chromosomes in pure species is a fundamental tenet of Mendelism. It has been extended to species hybrids involving the same and different (usually multiple) chromosome numbers, and has been made the basis of judgments as to phylogenetic relationships. Yet all such conclusions are valid only if the pairing at reduction is *invariably*

between chromosomes that are truly homologous; and this is clearly not the case. In two examples of crosses between tetraploids of known origin and diploid species of the same genus, non-homologous chromosomes have been found to pair. Obviously, conditions arise which promote the pairing of chromosomes which are not homologs. Possibly there is a variety of such causes. It is unwise to draw far-reaching conclusions as to the evolution of polyploid genera, therefore, on the basis of this single criterion.

[1] Ichijima, K., "Cytological and Genetic Studies on Fragaria," *Genetics*, **11**, 590–604 (1926).

[2] Gates, R. R., "The Behavior of Chromosomes in *Oenothera lata* × *gigas*," *Bot. Gaz.*, **48**, 179–199 (1909).

[3] Gates, R. R., "The Chromosomes of a Triploid Oenothera Hybrid," *Ann. Bot.*, **37**, 565–569 (1923).

CHROMOSOME BEHAVIOR IN SORBOPYRUS AND SORBARONIA

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In the Pomoideae subfamily of the family Rosaceae there are a number of interesting natural genus hybrids. Among these hybrids recognized by Rehder[2] are *Sorbaronia* (*Sorbus* × *Aronia*), *Sorbopyrus* (*Pyrus* × *Sorbus*) and *Amelasorbus* (*Amelanchier* × *Sorbus*). The relative frequence of such hybrids suggests that many of the genera in this subfamily may be rather closely related and that generic hybrids of horticultural value might be produced.

All three of the above generic hybrids are growing in the Arnold Arboretum but the plants of *Amelasorbus* are not large enough to produce flowers. The other two hybrids bloom freely and produce fruits each year. *Sorbaronia Dipelii* (*Sorbus Aria* × *Aronia melanocarpa*) is completely fertile and sets seeds abundantly. *Sorbopyrus auricularis bulbiformis* (*Pyrus communis* × *Sorbus Aria*) produces fruit but few seeds are developed. This year only five good seeds were obtained from more than fifty fruits.

Cytological studies on these hybrids were made to determine the chromosome compatibility and behavior. Permanent smears of the pollen mother cells were made which were fixed in Nawaschin's solution and stained with crystal violet iodine.

In *Sorbaronia Dippelii* p.m.c. there are 17 pairs of chromosomes at the first reduction division. Both reduction divisions are perfectly normal and all of the pollen grains appear to be functional. There are 17 pairs

of chromosomes in *Sorbus Aria*. Counts have not been obtained for *Aronia melanocarpa* but in view of the chromosome number and behavior in the hybrid both parents evidently have 17 gametic chromosomes which are completely compatible in the F_1 hybrid.

Sorbopyrus auricularis bulbiformis in the Arboretum is from scions sent from the Kew Gardens in England. This variety of *Sorbopyrus* is a seedling of *auricularis* and is more like the pear in fruit characters than the parental species.[3] A cytological study of the pollen mother cells shows that there are 17 paired and 17 single chromosomes at the first reduction division. Numerous counts show a total of 34 chromosomes at metaphase. In later stages of division 17 chromosomes can be counted at each pole and 17 dividing univalents between. The second division is also irregular and practically all of the pollen grains are aborted. *Sorbus Aria* has 17 gametic chromosomes and the species of *Pyrus* investigated by Dr. Nebel[1] also have 17 pairs of chromosomes. This variety of *Sorbopyrus* is evidently a back cross of a diploid egg cell of the F_1 hybrid with a haploid pollen grain from *Pyrus*. The presence of two sets of *Pyrus* chromosomes and one set of *Sorbus* chromosomes would explain why the variety is more like *Pyrus* than the parental hybrid. Evidently the *Sorbus* chromosomes will function in somatic development with either one or two sets of *Pyrus* chromosomes although the *Pyrus* and *Sorbus* chromosomes do not pair at the reduction divisions.

The chromosome behavior in these generic hybrids suggests that it may be necessary to make a taxonomic revision of the genera in the Pomoideae subfamily of the Rosaceae.

[1] Nebel, B., Zur Cytologie von Malus und Vitis. "Die Gartenbauwissenschaft," Band 1, *Heft.*, **6**, 449–592 (1929).

[2] Rehder, A., *Manual of Cultivated Trees and Shrubs*, The McMillan Co., New York, 1927.

[3] Tatar, M. "Pirus Bollwylleriana De. C. var. bulbiformis," *Wien. Obst-ud Gart. Zeitung.*, **3**, 26–28 (1878).

CHEMICAL ASPECTS OF DISEASE RESISTANCE IN THE ONION

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Striking differences in susceptibility to one or another parasite are commonly noted among closely related horticultural varieties of our economic plants. Inquiry into the causes of disease resistance is fundamental to permanent progress in the improvement of cultivated crop

plants and offers an important present day challenge to pathologist, geneticist and biochemist.

In a previous communication[1] the senior writer called attention to the striking degree of resistance in colored varieties of onion to the disease known as smudge. This is a disease which appears on the bulb, and the neck tissue just above the latter, shortly before maturity. The causal fungus, *Colletotrichum circinans* (Berk.) Vogl., persists from year to year in the soil, and infects the outer dry scales of the susceptible varieties during the last three or four weeks of the growing season. At harvest it is evident as black smudgy spots on these outer scales. Following harvest the fungus penetrates the cuticle of the underlying living fleshy scales and causes a gradual shrinkage and decay during curing and storage.

In the colored varieties exposed to the parasite in a similar manner, very little or no infection of the dry outer scales occurs, the only conspicuous symptoms being on the outer sheaths of the leaves above the bulb where comparatively little pigment develops. This last type of infection is of little economic consequence since the infected parts are largely removed at harvest. There is a close correlation of the absence of infection in the resistant varieties with the occurrence of red or yellow outer scale tissue. Where such pigmentation is poorly developed, as in the neck tissue above the bulbs or where the pigment has become bleached by unusual exposure to rainy weather, infection does occur, but as a rule further ingress of the parasite is checked by the next underlying tissue which is normally colored.

As pointed out in earlier papers,[1] the crude water extract from dry outer colored scales (yellow or red) is distinctly toxic to the spores and thalli of the fungus. This is readily demonstrated by placing either in a moderate concentration of the extract. Growth of thalli is immediately checked if not completely inhibited. The germinating spore shows an unusual response. The germ tube starts off in the regular manner but within a very short time the growing tips either dissolve or rupture, the exact process not having been determined as yet. The protoplasm of the spore is forced out through the small opening and it congregates characteristically as a naked mass alongside the exterior of the spore. In this way the spore is rendered functionless. In cold water extracts from outer scales of white varieties, on the other hand, normal germ tubes develop, the attachment bodies (appressoria) form, and from the latter infection tubes arise.

The basic reason for the marked resistance of colored bulbs to this disease seems to be due to the fact that in the scales of such bulbs there is formed a substance (or substances) not present in the scales of white bulbs, which is toxic to the fungus. Being soluble in cold water it diffuses out readily from the dead cells of the outer scales into drops or films of

meteoric or soil water in contact with the bulbs. It thus retards or kills the fungus within such drops of water before it can establish itself upon the host tissue.

The relatively simple manner in which the resistant quality of colored bulbs was brought into effect suggested the possibility of isolating the chemical compound or compounds responsible. At this point the junior authors joined in the investigation and undertook the major responsibility for the chemical study.

Regular qualitative tests with ferric chloride solution, dilute ammoniacal solution of potassium ferricyanide, methylene blue, iodine solution with ten per cent ammonia, and bromine water showed the presence of phenolic compounds in water extracts from red and from yellow scales, and their absence in extracts from white scales. This reaction of the colored extracts was to be expected because the pigment substances themselves are supposedly phenolic derivatives and those soluble in water would thus give a positive test. The yellow pigment, quercetin, which exists in the scales in the free state, is practically insoluble in cold water. Thus the reactions indicated that other phenolic compounds, more readily soluble in water, exist in red and yellow scales. On the other hand, even phenolic compounds that give a positive reaction with ferric chloride seem to be absent in white scales. The possibility of glucosides of quercetin or of tannin compounds being present was considered but all attempts to demonstrate them yielded negative results.[2]

Assuming that the toxic compounds were phenolic in character, the next procedure was to attempt their isolation without destruction of the toxic properties. Neutral solvents were used at temperatures below 60° C. Acetone was found to be the most efficient solvent. Pigmented scales were subjected to an exhaustive extraction with dry acetone. The acetone extract was concentrated to dryness under reduced pressure and the resulting residue of extracted matter taken up with water. This eliminated quercetin and also permitted a partial separation of inactive impurities. The water soluble portion produced the same effect upon the spores as did crude water extracts made directly from the scales. It was next concentrated under reduced pressure below 40°C. to a thick syrup and treated with a mixture consisting of about 20 times its volume of alcohol and 40 times its volume of ether. A reddish brown precipitate was filtered off, which when dissolved in water showed very little toxicity. However, when the alcohol-ether solution was concentrated to dryness and the resulting residue taken up in water and tested upon the spores it proved to be very toxic. These experiments indicated that the toxic principle might be separated in its pure form by using neutral solvents and temperatures below 60°C.

The next step was an attempt to secure a large quantity of the toxic

principle. A slightly different procedure was followed to effect this objective. About fifteen pounds, amounting to several bushels, of dry outer scales of yellow onions were steeped in water overnight. The amount of free quercetin thus removed was negligible. The colored aqueous extract was treated first with neutral, then with basic lead actates until no more precipitate was obtained. The latter was filtered off, suspended in a large volume of water and hydrogen sulphide passed in to decompose the lead salt. The lead sulphide was filtered off leaving the toxic substance in solution together with impurities. This was concentrated under reduced pressure at 40–50° until the syrup contained 80 to 85 per cent of solids. After standing for 12 hours a dark brown gum separated leaving a thinner, aqueous syrup which gave a strong positive test with a solution of ferric chloride. From this syrup the major portion of the toxic principle was removed by adding successive portions of cold 95 per cent alcohol and quickly removing the alcoholic liquor before the gummy precipitate went into solution.

The alcoholic extracts thus obtained were highly pigmented, usually a deep red. Most of this color-producing principle was removed by heating the alcoholic extract under a reflux condenser in the presence of successive small portions of activated blood charcoal. After three such treatments the alcoholic solution attained a light brown color and still gave a strong color reaction with a ferric chloride solution, which indicated that the active substance, or substances, were still present. The alcohol was removed under reduced pressure at 35°. The yellow syrup obtained was taken up in warm 20 per cent alcohol, and warm petrol ether, b. p. 60–80° (1 part to 5) was added. The solution was then placed in a desiccator over calcium chloride. After three days, crystals formed on the sides of the vessel containing the solution. These crystals were removed, dried in a vacuum at 95°, and on purification were found to be pale yellow needles with a constant melting point of 199°. The substance was identified as the phenolic acid known as protocatechuic acid (3,4-dihydroxybenzoic acid), by preparing from it the ethyl and methyl esters and the diacetyl derivative, all of which exhibited melting points identical with the accepted constants for these substances. As a final criterion the ultimate analysis indicated that the isolated acid had the elementary composition of carbon and hydrogen in agreement with the theoretical values required by the formula $C_7H_6O_4$.[3]

The evidence points to the conclusion that this acid is one of the chief toxic substances which make up the disease-resistant principle of colored onion. It was prepared with equal facility from yellow and from red varieties and repeated attempts yielded none from the white susceptible varieties. Furthermore, at dilutions of 1–2000 to 1–3000 in distilled water it caused the typical rupturing of the spores of the parasite already de-

scribed as a characteristic property of the crude aqueous extract from colored scales. It should also be noted that the toxicity of the pure protocatechuic acid isolated from onion scales is identical with the toxicity of the pure acid from other sources, East Indian kino for instance. This fact serves as a criterion of biological purity and eliminates the possibility of the protocatechuic acid isolated from the onion scales being contaminated with minute traces of another very potent toxic entity.

The phenolic acid, protocatechuic acid, is widely distributed in plants as a constituent of many aromatic compounds, in the catechol tannins, in numerous resins and wood gums, in lignified wood, and as a constituent of various flavone and anthocyan pigments. Its occurrence in the free state has, however, been reported in only a few cases. It is significant that the flavonol pigment quercetin has long since been recognized to be one of the pigments in the colored onion. Protocatechuic acid is a constituent of the quercetin molecule. On alkaline fusion, quercetin yields oxalic acid, phloroglucinol and protocatechuic acid. The close association of protocatechuic acid to the pigment quercetin deserves emphasis but it should be borne in mind that protocatechuic acid itself is not a pigment. The isolation of protocatechuic acid from pigmented onion scales represents to our knowledge the third instance that this acid has been found associated with the flavonol quercetin.

Since the isolation of the acid was performed by the application of chemical methods that exclude the possibility of the acid having arisen from quercetin by decomposition, it is definitely established that protocatechuic acid exists in the free state, and that it is one of the toxic principles involved in the resistance exhibited by the pigmented onion to the smudge disease.

It should be pointed out that the toxicity of the crude aqueous extract is greater than the toxic effects that could be ascribed to the amount of protocatechuic acid that was isolated from a given unit of toxic extract. It is, therefore, possible that additional toxic substances are present or that all of the acid was not isolated by the methods employed. The investigations are being extended, therefore, with the purpose of isolating other toxic substances and if possible to increase the yield of protocatechuic acid.

To our knowledge this is the first time that disease resistance in plants has been attributed to a definite chemical entity present in the resistant host (the pigmented onions) and absent in the non-resistant host (the white onions).

[1] See J. C. Walker, "Studies on Disease Resistance in the Onion," *Proc. Nat. Acad. Sci.*, **11**, 183–189 (1925); also "Disease Resistance to Onion Smudge," *J. Agr. Research*, **24**, 1019–1040 (1923).

[2] The statement previously made (*Proc. Nat. Acad. Sci.*, **11**, 188) that quercetin

exists in the onion combined with glucose is therefore erroneous. Perkin and Hummel (*J. Chem. Soc.*, **69**, 1295–1298) also state they failed to find a glucoside of quercetin in onion.

[2] For further details regarding the isolation and identification of the acid, see Link, Angell and Walker, "The Isolation of Protocatechuic Acid from Pigmented Onion Scales and Its Significance in Relation to Disease Resistance in Onions," *J. Biol. Chem.*, **81**, 369–375 (1929); and Angell, Walker and Link, "A Chemical Compound Responsible for Disease Resistance in the Onion," *J. Agr. Research* (1929). In press.

ON THE EXISTENCE OF INTEGRALS OF THE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS $A^i_{\alpha\beta i} = 0$ IN n VARIABLES

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§1. In this note we consider the integrals $\Gamma^i_{\alpha\beta}(=\Gamma^i_{\alpha\beta})$ of the system of equations

$$A^i_{\alpha\beta i} = 0 \qquad \text{(a)}$$

where the quantities $A^i_{\alpha\beta\gamma}$ are the components of the affine normal tensor.[1] Since the components $A^i_{\alpha\beta\gamma}$ are definite linear expressions in the components of affine connection $\Gamma^i_{\alpha\beta}(x)$ and their first derivatives, equations (a) constitute a system of linear partial differential equations of the first order in a set of unknown functions $\Gamma^i_{\alpha\beta}$. These equations are, in fact, equivalent to the system of equations obtained by setting the contracted affine curvature tensor $\beta^i_{\alpha\beta i}$ equal to zero, i.e.,

$$\frac{\partial\Gamma^i_{\alpha\beta}}{\partial x^i} - \frac{\partial\Gamma^i_{\alpha i}}{\partial x\beta} + \Gamma^\sigma_{\alpha\beta}\Gamma^i_{\sigma i} - \Gamma^\sigma_{\alpha i}\Gamma^i_{\sigma\beta} = 0.$$

We use the normal tensor $A^i_{\alpha\beta\gamma}$ instead of the equivalent curvature tensor $\beta^i_{\alpha\beta\gamma}$ as the former is more readily adapted to most theoretical investigations. When reference is made to a system of normal coördinates y^i instead of to arbitrary coördinates x^i, the designation $C^i_{\alpha\beta}(y)$ instead of $\Gamma^i_{\alpha\beta}(x)$ will be used. We first treat the case $n = 2$ for which the existence of integrals $\Gamma^i_{\alpha\beta}$ or $C^i_{\alpha\beta}$ of the system (a) is immediate when use is made of one of the results of an earlier article (§3).[2] The general case $n \geqq 3$ is then treated (§4) and a corresponding existence theorem is established. We have not given the convergence proof for the formal power series expansions of the components $A^i_{\alpha\beta\gamma}$ for $n \geqq 3$ as it is intended to publish this later as part of a systematic account of the existence of integrals of systems of tensor equations of the above type (a).

§2. The independent components of the normal tensor $A^i_{\alpha\beta\gamma}$, i.e., those components which can be taken as arbitrary after the conditions[3]

$$A^i_{\alpha\beta\gamma} = A^i_{\beta\alpha\gamma};\ A^i_{\alpha\beta\gamma} + A^i_{\beta\gamma\alpha} + A^i_{\gamma\alpha\beta} = 0 \qquad \text{(b)}$$

have been imposed, can be divided into mutually distinct groups G_m by the following

RULE.[2] *The group G_m ($m = 0, 1, \ldots, n-2$) for the components $A^i_{\alpha\beta\gamma}$ is composed of all components that can be formed from $A^i_{\alpha\beta\gamma}$ by taking $\gamma = m+1$; $i = 1, \ldots, n$; $\alpha, \beta = 1, \ldots, n$ subject to the inequalities $\alpha \leqq \beta$, and $\beta > m+1$.* The number A_m of components $A^i_{\alpha\beta\gamma}$ in group G_m is equal to $nK(n, 2) - nK(m+1, 2)$ where $K(p, q)$ is used to denote the number of combinations with repetitions of p things taken q at a time. For simplicity let us introduce the designation A_{lm} ($l = 1, \ldots, A_m$) for the components $A^i_{\alpha\beta\gamma}$ in group G_m. It can then be shown that a system of equations of the form

$$\frac{\partial A_{lm}}{\partial x^k} = \sum \frac{\partial A_{pq}}{\partial x^r} + \bigstar \qquad \text{(c)}$$

$$\left(\begin{array}{c} m = \mu,\ 1 \leqq \mu \leqq n-2 \\ l = 1, \ldots, A_\mu \\ k = 1, \ldots, \mu \end{array}\right)$$

must be satisfied, where $\sum$ represents a linear expression in the derivatives $\partial A_{pq}/\partial x^r$, and the $\bigstar$ terms denote polynomials in the A_{lm} and the components of affine connection. Also the derivatives $\partial A_{pq}/\partial x^r$ in the right member of any equation (c) for which the index m has the value μ are such that the inequalities $q \leqq \mu$, $r > q$ are satisfied.[2]

Lemma. The contravariant index i has the same value for all components $A^i_{\alpha\beta\gamma}$ whose derivatives occur in any particular equation (c). This fact which is obvious from the method of derivation of equations (c) will be used in §4.

On account of (b) there are only $K(n, 2)$ independent equations (a). We shall agree to consider, for definiteness, only those equations (a) for which the condition $\alpha \leqq \beta$ is satisfied.

§3. It has been shown that each of the components A_{lo} forming the group G_0 can be taken as an arbitrary analytic function of a set of n variables y^i which can be identified with the coördinates of a system of normal coördinates.[2] In case $n = 2$ all independent components $A^i_{\alpha\beta\gamma}$ belong to group G_0. Hence if we write equations (a) in the expanded form

$$A^2_{121} = 0;\ 2A^1_{121} - A^2_{221} = 0;\ A^1_{221} = 0,$$

we have the

Existence Theorem A. Let $F(y)$ denote an arbitrary function of the

variables y^1, y^2 analytic in the neighborhood of the values $y^1 = y^2 = 0$. Then there exists one, and only one, affine connection with components $C^i_{\alpha\beta}$ $(= C^i_{\beta\alpha})$ in a system of normal coördinates y^i, each function $C^i_{\alpha\beta}$ (y) being analytic in the neighborhood of the values $y^1 = y^2 = 0$, which constitutes a set of integrals of the system of equations (a) *and which is such that the component A^1_{121} (or A^2_{221}) of the resulting normal tensor is identically equal to the function $F(y)$.*

To extend this theorem to arbitrary coördinates x^i we consider a set of functions $Z^i(x)$ defined by the power series expansions

$$Z^i(x) = \sum_{s=2}^{\infty} (1/s!)(\Gamma^i_{\sigma_1 \ldots \sigma_s})_q (x^{\sigma_1} - q^{\sigma_1}) \ldots (x^{\sigma_s} - q^{\sigma_s}), \qquad \text{(d)}$$

in which the constants q^i are arbitrary and the coefficients Γ are subject only to the condition that the series converge. Take the component A^1_{121} (or A^2_{221}) to be an arbitrary function Ψ (x) analytic in the neighborhood of the point $x^i = q^i$. All other components $A^i_{\alpha\beta\gamma}$ (x) will then be determined by equations (a) and (b). Now transform the components $A^i_{\alpha\beta\gamma}$ (x) into a set of components $\tilde{A}^i_{\alpha\beta\gamma}$ (y) by the transformation

$$x^i = q^i + y^i - \sum_{s=2}^{\infty} (1/s!)\ (\Gamma^i_{\sigma_1 \ldots \sigma_s})_q\ y^{\sigma_1} \ldots y^{\sigma_s}. \qquad \text{(e)}$$

Since the expressions $\tilde{A}^i_{\alpha\beta i}$ must vanish identically as functions of the variables y^i the above theorem states that there exists an affine connection with components $C^i_{\alpha\beta}$ (y) which depend on a set of normal coördinates y^i and which are such that the components of the resulting normal tensor, as functions of the variables y^i, are identically equal to the functions $\tilde{A}^i_{\alpha\beta\gamma}$ (y). Now transform the components $C^i_{\alpha\beta}$ (y) into a set of components $\Gamma^i_{\alpha\beta}$ (x) by the transformation (e). The above functions $A^i_{\alpha\beta\gamma}$ (x) will then be the components of a normal tensor with respect to the $\Gamma^i_{\alpha\beta}$ (x); also at $x^i = q^i$ the components $\Gamma^i_{\alpha\beta}$ (x) and the components of the generalized connections $\Gamma^i_{\alpha\beta \ldots \epsilon}$ (x) which are determined by $\Gamma^i_{\alpha\beta}$ (x) will assume the values of the corresponding coefficients Γ in the expansions (d).[4] This proves the following:

Existence Theorem B.—Let $Z^i(x)$ denote a set of two arbitrary functions defined by (d); *also let Ψ (x) denote an arbitrary function of the variables x^1, x^2 analytic in the neighborhood of the values $x^i = q^i$. Then there exists one, and only one, affine connection with components $\Gamma^i_{\alpha\beta}$ $(= \Gamma^i_{\beta\alpha})$, each function $\Gamma^i_{\alpha\beta}$ (x) being analytic in the neighborhood of the values $x^i = q^i$, which constitutes a set of integrals of the system of equations* (a) *and which is* (1) *such that at $x^i = q^i$ the components of affine connection $\Gamma^i_{\alpha\beta}$ and of the generalized connections $\Gamma^i_{\alpha\beta \ldots \epsilon}$ assume the values of the corresponding coefficients in the power series expansions* (d) *of the functions Z^i (x) and* (2) *such that*

the component A^1_{121} (or A^2_{221}) of the resulting normal tensor is identically equal to the function $\Psi(x)$.

§4. In case $n \geqq 3$ let us observe that the left member of any equation (a), supposing $\alpha \leqq \beta$, contains components $A^i_{\alpha\beta\gamma}$ which belong to groups G_0 to $G_{\beta-1}$, inclusive. An equation (a), for which α and β are not both equal to 1, contains the component $A^1_{\alpha\beta_1}$ which is in group G_0; if $\alpha = \beta = 1$ in (a) the equation contains the component A^2_{121} belonging to G_0. Let us denote these $K(n, 2)$ specified components which belong to group G_0 by A_{l_n} ($l = 1, \ldots, K(n, 2)$). We denote the remaining components in the group G_0 by A_{l_0} where $l = 1, \ldots, A_0 - K(n, 2)$. As it is easily seen that a component A_{l_n} occurs in one, and only one, equation (a) we can solve the equations (a) for the components A_{l_n} by a simple transposition of the remaining terms. A subsequent differentiation with respect to the variable x^k then gives a system of equations having the form

$$\frac{\partial A_{l_n}}{\partial x^k} = \sum \frac{\partial A_{uv}}{\partial x^k} \qquad \text{(f)}$$

$$\begin{pmatrix} l = 1, \ldots, K(n, 2) \\ k = 1, \ldots, n \end{pmatrix}.$$

Now eliminate, by the substitution (c), those derivatives in the right members of (f) which occur in the left members of (c). Let us refer to the resulting equations as the system of equations R_1. For convenience of reference we shall furthermore divide the system R_1 into two separate systems, namely, the system S_1, composed of those equations of R_1 which contain derivatives of the component A^2_{121} in their left members, and the system T_1 composed of the remaining equations of R_1. Since the A_{uv} whose derivatives occur in the right member of an equation (f) denote components $A^i_{\alpha\beta\gamma}$ for which the contravariant index i is different from the corresponding index of the component whose derivative forms the left member of the equation, it follows by the above lemma that the derivative in the left member of an equation of the system R_1 will not occur in the right member of the same equation. In fact, only those derivatives $\partial A_{l_n}/\partial x^k$ can occur in the right members of R_1 which correspond to derivatives of the component A^2_{121}. Derivatives of the component A^2_{121} can, moreover, occur only in the right members of the system T_1 and it is easily seen that these can be eliminated by a substitution involving the equations of the system S_1. As the result of this latter substitution the system of equations R_1 becomes a system R_2. Finally we use the equations of the system R_2 to eliminate the derivatives $\partial A_{l_n}/\partial x^k$, which constitute the left members of R_2, from the right members of equations (c); the equations which result from (c) will be said to form the system R_3. On com-

bining the two systems R_2 and R_3 we obtain a system of equations of the form

$$\frac{\partial A_{lm}}{\partial x^k} = \sum \frac{\partial A_{pq}}{\partial x^r} + \bigstar \qquad \text{(g)}$$

$$\left(\begin{matrix} m = \mu;\ \mu = 1, \ldots, n-2;\ \mu = n \\ l = 1, \ldots, A_\mu \text{ where } A_n = K(n, 2) \\ k = 1, \ldots, \mu \end{matrix} \right),$$

for which the derivatives in the left members do not occur in the right member of any equation of the system. It should be observed that $0 \leqq q \leqq n-2$, that $1 \leqq r \leqq n$, and that $r > q$ for any derivative in the right member of an equation (g).

Now denote by

$$\Psi_{lm} \left(\begin{matrix} l = 1, \ldots, A_0 - K(n, 2) \text{ for } m = 0; \\ l = 1, \ldots, A_m \text{ for } m = 1, \ldots, n-2 \end{matrix} \right) \qquad \text{(h)}$$

an arbitrary function of the variables $y^{m+1}, y^{m+2}, \ldots, y^n$ which is analytic in the neighborhood of the values $y^{m+1} = \ldots = y^n = 0$. (Variables y^i instead of x^i are now used since they will later be identified with the coördinate of a system of normal coördinates.) Put $A_{lo} = \Psi_{lo}$; also put $A_{lm} = \Psi_{lm} (m \neq 0)$ for $y^1 = \ldots = y^m = 0$. These components A are therefore determined at $y^i = 0$; from these components A, equations (a), and equations (b), the remaining components A_{lm} at $y^i = 0$ are determined. If we denote by $(A_{lm,\sigma_1 \ldots \sigma_p})_0$ a derivative of the component A_{lm} of order p evaluated at $y^i = 0$, then the number of pth order derivatives of the functions Ψ_{lm}, i.e., the number of *arbitrary* quantities $(A_{lm,\sigma_1 \ldots \sigma_p})_0$ is given by the expression

$$[A_0 - K(n, 2)]\, K(n, p) + A_1 K(n-1, p) + \ldots + A_{n-2} K(2, p) \qquad \text{(i)}$$

The remaining derivatives $(A_{lm,\sigma_1 \ldots \sigma_p})_0$ can be found from (g) and from those equations obtained from (g) by successive differentiation.[2] Moreover, any two determinations of one of these latter derivatives $(A_{lm,\sigma_1 \ldots \sigma_p})_0$, for example, by making a variation in the order of differentiation of equations (g), must result in the same value of the derivatives $(A_{lm,\sigma_1 \ldots \sigma_p})_0$. If this were not the case it would be necessary as a condition of integrability for the number of arbitrary quantities $(A_{lm,\sigma_1 \ldots \sigma_p})_0$ to be less than that given by (i). Since the determination of $K(n, 2)K(n, p)$ of the quantities $(A_{lm,\sigma_1 \ldots \sigma_p})_0$ is due to the existence of conditions (a), this would mean that the number of arbitrary quantities $(A_{lm,\sigma_1 \ldots \sigma_p})_0$ in the general affinely connected space would be less than

$$A_0 K(n, p) + A_1 K(n-1, p) + \ldots + A_{n-2} K(2, p)$$

or $N(n, 1 + p)$ which is impossible.[5] Hence the power series expansion of an independent component A_{lm} has a unique determination; and as any dependent component $A^i_{\alpha\beta\gamma}$ is, in accordance with (b), a simple linear expression in the independent components A_{lm}, the determination of the power series expansion of any component $A^i_{\alpha\beta\gamma}$, namely,

$$A^i_{\alpha\beta\gamma} = (A^i_{\alpha\beta\gamma})_0 + (A^i_{\alpha\beta\gamma,\sigma_1})_0\, y^{\sigma} + 1/2!\,(A^i_{\alpha\beta\gamma,\sigma_1\sigma_2})_0 \qquad \text{(j)}$$

$$y^{\sigma_1}y^{\sigma_2} + \ldots,$$

is unique. The convergence of the expansions (j) will, as we have already said, be given in a later publication. It can then be shown that there exists an affine connection with components $C^i_{\alpha\beta}$ (y) with respect to which (1) the functions $A^i_{\alpha\beta\gamma}$ defined by (j) are the components of the normal tensor and (2) the variables y^i are the coördinates of a system of normal coordinates.[2] As these components $C^i_{\alpha\beta}$ (y) obviously constitute a set of integrals of (a) we have the following

Existence Theorem C. Let Ψ_{lm} with range of indices l, m specified by (h), *denote an arbitrary function of the variables y^{m+1}, y^{m+2}, $\ldots$ y^n analytic in the neighborhood of the values $y^{m+1} = \ldots = y^n = 0$. Then there exists one, and only one, affine connection with components $C^i_{\alpha\beta}$ $(= C^i_{\beta\alpha})$ in a system of normal coördinates y^i, each function $C^i_{\alpha\beta}$ (y) being analytic in the neighborhood of the values $y^i = 0$, which constitutes a set of integrals of the system of equations* (a) *and which is such that the components A_{lm} of the resulting normal tensor are*

$$A_{lo} = \Psi_{lo}(l = 1, \ldots, A_0 - K(n, 2)); \text{ and}$$

$$A_{lm} = \Psi_{lm}(l = 1, \ldots, A_m; m = 1, \ldots, n - 2)$$

for $y^1 = \ldots = y^m = 0$.

The extension of this theorem corresponding to Existence Theorem B will be considered in the article above proposed.

[1] O. Veblen, "Normal Coördinates for the Geometry of Paths," *Proc. Nat. Acad. Sci.*, **8**, 192 (1922). See also, O. Veblen and T. Y. Thomas, "The Geometry of Paths," *Trans. Am. Math. Soc.*, **25**, § 9, 566 (1923).

[2] T. Y. Thomas, "The Existence Theorems in the Problem of the Determination of Affine and Metric Spaces by Their Differential Invariants." Not yet published.

[3] T. Y. Thomas, "The Identities of Affinely Connected Manifolds," *Math. Zeit.*, **25**, 714(1926).

[4] T. Y. Thomas, "A Theorem Concerning the Affine Connection," *Am. J. Math.*, **50** 518(1928).

[5] See footnote to § 3, loc. cit. (2).

CLOSED EXTREMALS

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The object of this paper is to present briefly some of the principal results of a theory of closed extremals.

We are concerned with a calculus of variations problem in the ordinary parametric form with an integrand which is positive, analytic, homogeneous in the usual way, and positively regular. See reference 3, ¶2. The domain of the coördinates is to be a regular, analytic, orientable $(m-1)$-dimensional manifold R in m-space.

We classify closed extremals g according to their type numbers as follows.

We start by mapping g and its neighborhood on an x-axis and its neighborhood in a space $(x, y_1, \ldots, y_n)$, using thereby a transformation which has a period in x equal to the length ω of g. Let γ denote the portion of the x axis for which $o \leqq x \leqq \omega$.

We cut across γ by $p + 1$ successive n-planes, $t_0, \ldots, t_p$, of which the first is $x = o$, and the last $x = \omega$, and which are placed so near together that there are no pairs of conjugate points on the successive segments into which γ is divided. Let P_i be any point on t_i near γ, except that P_o and P_p shall differ only in their x coördinates. The points $P_o, \ldots, P_p$, can be successively joined by extremal segments neighboring γ. Let the resulting broken extremal be denoted by E. Let (u) be the set of $\mu = pn$ variables, of which the first n are the coördinates (y) of P_o, the second n those of P_s, and so on, until finally the last n are the coördinates (y) of P_{p-e}. The value of the transformed integral taken along E will be a function of (u) and will be denoted by $J(u)$.

The function $J(u)$ will have a critical point when $(u) = (o)$. We consider next the form

$$Q = \sum_{ij} J_{u_i u_j} u_i u_j \qquad (i, j = 1, \ldots, \mu), \tag{1}$$

where the partial derivatives are evaluated for $(u) = (o)$. We classify our closed extremals according to the type number and the nullity of the form Q.

THEOREM 1. *The nullity of the form Q equals the number of linearly independent solutions of the Jacobi differential which have the period ω.*

To aid in determining the type number of Q we first define the order of concavity of γ.

If $x = o$ on γ is not conjugate to $x = \omega$ on γ, any of the above points

P_o can be joined to its congruent point P_p by a secondary extremal.* Let P_o be restricted to points on $x = o$ for which

$$y_1^2 + \ldots + y_n^2 = 1.$$

The resulting set of secondary extremals joining P_o to P_p will form a sort of tubular manifold M roughly enclosing γ. The exterior normals to M at P_o and P_p will be in a 2-plane through the x axis. Let θ be the angle between these normals, measuring θ from the normal at P_o, and counting that sense of rotation positive which leads from the positive x axis to either of these normals. We have

$$r(y) \sin \theta = \sum_{ij} a_{ij}\, y_i y_j \qquad (i, j = 1, \ldots, n), \tag{2}$$

where (y) gives the coördinates of P_o, $r(y)$ is a positive continuous function of (y), and the coefficients a_{ij} are constants.

The type number of the right-hand form in (2) will be called the order of concavity of γ.

THEOREM 2. *If $x = o$ is not conjugate to $x = \omega$ on γ, the type number of Q will equal the sum of the orders of the conjugate points of $x = o$ preceding $x = \omega$, plus the order of concavity of g.*

As an example consider the $(m\text{-}1)$-dimensional ellipsoid

$$\frac{x_1^2}{a_1^2} + \ldots + \frac{x_m^2}{a_m^2} = 1 \qquad o < a_1 < \ldots < a_m. \tag{3}$$

The ellipse lying in the coördinate 2-plane in which all the x's are zero except x_i and x_j, $i \neq j$ will be called the ijth principal ellipse.

THEOREM 3. *The type number of the ijth principal ellipse is $m + i + j - 5$.*

Thus in 3-space the type numbers are 1, 2 and 3. For a general m the total number of principal ellipses is ${}_mC_2$, the number of combinations of m objects two at a time.

The author has developed a general deformation theory in m-space, extending and completing the theory already developed in the small in the paper 2 cited below. In this theory the following is a fundamental theorem.

THEOREM 4. *Suppose g is non-singular of type $k > o$, and gives to J the value c.*

Then corresponding to a sufficiently small neighborhood N of g, there exists within N an arbitrarily small neighborhood N' of g, such that closed m-families of curves which are deformable into g on N, and which satisfy

$$J \leqq c - e, \tag{4}$$

where e is a sufficiently small positive constant, are conditioned as follows.

(a) *Those for which* $m \neq k\text{-}1$ *can be deformed on* N *without increasing* J *into a family of lower dimensionality.*

(b) *Those for which* $m = k\text{-}1$ *include a* $(k\text{-}1)$*-family* Z *which is non-bounding on* N *and* (4)*, and which is such that every other* $(k\text{-}1)$*-family can be deformed on* N *without increasing* J *either into* Z *or else a single closed curve.*

More generally still one replaces each sensed closed curve on R on which J is less than a constant J_o, by a succession of q short extremals (q large), joining successive vertices on the given closed curve. The set of vertices on the given curve furnish a point (π) in an auxiliary domain Σ. Two points (π) are to be regarded as identical if the corresponding sets of vertices define the same sensed broken extremal. Particular attention is paid to the reversible problem, that is, the problem in which a point and initial direction, and the same point and opposite direction define the same extremal. In the reversible problem two points (π) are regarded as identical if they define the same broken extremal regardless of sense.

The relation between the connectivities of the domain Σ and the type numbers of closed extremals for which $J < J_o$ is illustrated by the following typical theorem.

THEOREM 5. *Corresponding to an increase of the constant* J_o *through a value of* J *taken on by a single, non-singular, closed extremal of type* k *in a reversible problem, either the* k*th connectivity of the domain* Σ *increases by one, or else the* $(k\text{-}1)$*st connectivity decreases by one.*

To come to a particular case of great importance let R be the topological image of an $(m\text{-}1)$-sphere under a correspondence in which great circles are carried into closed curves of length J less than J_l.

THEOREM 6.[5] *On a manifold* R *which is the topological image of an* $(m\text{-}1)$*-sphere, and under a proper count of multiplicities, there will always exist at least* ${}_mC_2$ *closed geodesics on which* $J < J_l$. *If these closed geodesics are non-singular they will be distinct, and of the respective types of the coordinate ellipses on an* $(m\text{-}1)$*-dimensional ellipsoid.*

A specialization of the preceding methods leads to interesting results in the theory of diameters, that is chords which are orthogonal to a manifold at both ends.

In the first place the length of the chord may be regarded as a function F of the parameters which specify its end-points. The function F will have a critical point if and only if the chord is a diameter. At a critical point the diameter will be said to have the type number of F at that point, and to be non-singular if that critical point is non-singular.

For example the ellipsoid has a diameter on each axis, and the type number of the diameter on the ith axis equals $m + i - 2$. Corresponding to these facts we have the following existence theorem in general.

THEOREM 7. *A regular manifold which is a topological image of an*

(m-1)-sphere possesses at least m diameters, counting multiplicities properly. If these diameters are non-singular they will be distinct, and of types m-1 to 2(m-1), respectively.

The method of proof is to set up explicitly the non-bounding circuits in the domain of the variables in F, and to prove the non-existence of homologies between them.

* A curve representing a solution of the Jacobi differential equations.

[1] "Relations Between the Critical Points of a Real Function of n Independent Variables," *Trans. Amer. Math. Soc.*, **29** (1927), pp. 429–463.

[2] "The Foundations of a Theory in the Calculus of Variations in the Large," *Ibid.*, **30** (1928), pp. 213–274.

[3] "The Foundations of the Calculus of Variations in the Large in m-Space," (first paper) *Ibid.*, **31** (1929), pp. 379–404.

[4] "The Critical Points of Functions and the Calculus of Variations in the Large," *Bull. Amer. Math. Soc.*, **35** (1929), pp. 38–54.

[5] For a proof of the existence of three closed geodesics on a convex surface, and of at least one closed geodesic on R in the m-dimensional case see Birkhoff, "Dynamical Systems," *Amer. Math. Soc., Colloquium Publications*, **9**, 1927.

NON-ABELIAN GROUPS OF ODD PRIME POWER ORDER WHICH ADMIT A MAXIMAL NUMBER OF INVERSE CORRESPONDENCIES IN AN AUTOMORPHISM

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It is known that a necessary and sufficient condition that a given group G is abelian is that every one of its operators corresponds to its inverse in one of the possible automorphisms of G, and that at most three-fourths of the operators can correspond to their inverses in an automorphism of a non-abelian group. Moreover, a non-abelian group cannot admit such an automorphism unless its order is divisible by 8 and it satisfies certain other conditions.[1] In particular, a non-abelian group whose order is of the form p^m, p being an odd prime number, can therefore not admit an automorphism in which as many as three-fourths of its operators correspond to their inverses. In the present article we shall prove that not more than p^{m-1} of the operators of such a group can correspond to their inverses in an automorphism thereof, and that there are such groups in which exactly p^{m-1} operators correspond to their inverses in one of the possible automorphisms for every odd prime value of p and for every value of $m > 2$.

Let G represent any non-abelian group of order p^m, p being an odd

prime number, which satisfies the condition that a maximal number of its operators correspond to their inverses in an automorphism thereof, and suppose that such an automorphism of G has been established. Let H represent a maximal subgroup among those which satisfy the condition that each of their operators corresponds to its inverse in this automorphism. It is well known that H is an abelian group and that it is invariant under a group K of order $k = hp$, h being the order of H. To prove that no operator of K except those of H corresponds to its inverse in this automorphism it is only necessary to observe that if s and t are two operators of H and K, respectively, such that $t^{-1}st = s_0 s$ and $t \sim t^{-1}$, then we may assume that s_0 is not the identity for every value of s since otherwise H would not be a maximal subgroup of G in which every operator corresponds to its inverse. That is, $st \neq ts$, and we may also assume without loss of generality that s_0 is commutative with t. Hence it results that $ts^{-1}t^{-1} = s_0 s^{-1}$. This evidently contradicts the conditions: $s_0 \sim s_0^{-1}$, $s \sim s^{-1}$, $t \sim t^{-1}$. Hence it follows that no operator of K except those found in H corresponds to its inverse in the given automorphism of G.

From the preceding paragraph it also results that if r is any other operator of G which corresponds to its inverse in the given automorphism then r cannot transform H into itself and hence it cannot be commutative with more than h/p of the operators of H. Moreover, from $rs \sim s^{-}r^{-1}$, it results that r and s are commutative, since rs also corresponds to $r^{-1}s^{-1}$. Hence it results that when the operators of G are arranged in co-sets with respect to H then not more than $1/p^{th}$ of the operators of a co-set can correspond to their inverses, and there are at least $p-1$ co-sets in which no operator corresponds to its inverse in the given automorphism of G. Hence we have established the following theorem: *A non-abelian group of odd prime power order cannot admit an automorphism in which more than $1/p^{th}$ of its operators correspond to their inverses.*

To prove that there is at least one non-abelian group of order p^m, $m > 2$, which admits an automorphism in which p^{m-1} of its operators correspond to their inverses, it is only necessary to note that when a non-abelian group G of order p^m contains an abelian subgroup H of order p^{m-1} and an additional operator t such that $t^{-p} = (st)^p$ then t may correspond to st in the given automorphism. This condition is evidently always satisfied when $t^p = s = 1$. It should be noted that when p^{m-1} of the operators of G correspond to their inverse in an automorphism of G these operators do not necessarily constitute a subgroup of G. When they do not form such a subgroup they obviously generate G and hence the automorphism is necessarily of order 2, while the automorphism may be of order $2p$ when these operators constitute a subgroup of G.

Suppose that G admits an automorphism in which p^{m-1} of its operators correspond to their inverses but these operators do not constitute a sub-

group of G. Assuming that H has the same meaning as in the second paragraph it results that H is invariant under exactly hp of the operators of G. Moreover, each of the remaining operators of G which corresponds to its inverse in the given automorphism is commutative with h/p of the operators of H. We proceed to prove that these h/p operators are the same for every operator of G which corresponds to its inverse, and that they therefore appear in the central of G. It may first be noted that if r is any operator of G which corresponds to its inverse in the given automorphism and transforms K into itself, and if t is any operator of K which is not also in H while s is an operator of H with which r is non-commutative then $r^{-1}sr$ cannot appear in H and hence $t^{-1}r^{-1}srt$ can also not appear in H. That is, rt is not commutative with s unless r has this property. If rt is an operator which corresponds to its inverse in the given automorphism it must be commutative with the same operators of H as r is and hence H involves a subgroup of order h/p which is in the central of the group generated by r and K. We may assume that this group is of order hp^2.

As H is transformed into itself only by hp operators of G it results that under the group of order hp^2 generated by K and r, all the conjugates of H appear in K and these conjugates involve only one subgroup in which all the operators correspond to their inverses. If the order of G would exceed hp^2 and if μ would be an operator of G which corresponds to its inverse in this automorphism but is not contained in this subgroup of order hp^2 it would result as before that μ would be commutative with the same h/p operators of H which appear in the central of K. This is impossible since the quotient group of the group generated by r and K with respect to the said invariant subgroup of order h/p contained in H is the non-abelian group of order p^3 which involves no operator of order p^2. Each of the $p + 1$ subgroups of order p^2 involves one and only one subgroup of order p whose operators correspond to their inverses in the given automorphism. Moreover, p of the subgroups of order p contained in these subgroups of order p^2 are conjugate under the group generated by r and K. As one of these subgroups of order p^2 would be invariant under a quotient group of order p^4 if the order of G would exceed hp^2 the subgroup satisfying the same condition as H satisfies would have to be transformed into itself under G by more than hp operators. Hence the following theorem has been established: *If a group of order p^m, p being an odd prime number, admits an automorphism in which exactly p^{m-1} operators correspond to their inverses but these operators do not form a subgroup thereof, then they must appear in $p + 1$ abelian subgroups of order p^{m-2} which have in common a characteristic subgroup of order p^{m-3}.*

It is easy to prove that there is at least one group of order p^m, $m > 2$, which admits an automorphism in which exactly p^{m-1} of its operators correspond to their inverses and these operators generate the entire group.

In fact, the non-abelian group of order p^m which contains $p + 1$ abelian subgroups of order p^{m-1} and involves no operator of order p^2 evidently satisfies this condition.

In such a group H may be a non-invariant abelian subgroup of order p^{m-2} while K is the abelian subgroup of index p which includes this subgroup. For r we may then select any operator of the group which is not found in this subgroup of index p. Hence it results that whenever $m > 2$ it is possible to construct a non-abelian group of order p^m in which exactly p^{m-1} operators which generate the entire group correspond to their inverses, and it is also possible to construct such a group in which exactly p^{m-1} operators which appear in a subgroup thereof correspond to their inverses. These groups are evidently not necessarily completely determined by the fact that they satisfy the given conditions.

Both of the non-abelian groups of order, p^3, p being an odd prime, evidently have the property that p^2 of their operators correspond to their inverses in a possible automorphism of the group, but in only one of these is it possible to select these operators so that they generate the entire group, viz., in the one which involves no operator of order p^2. Moreover, only the p cyclic subgroups of order p^2 in the other have the property that each of their operators can correspond to its inverse in an automorphism of this group since it is evidently impossible to make all the operators of the non-cyclic subgroup of this order correspond to their inverses in such an automorphism. In closing we may direct attention to a dissimilarity existing between the non-abelian groups of order p^m, p being an odd prime, which have the property that a relatively maximal number of their operators correspond to their inverses in one of their possible automorphisms and the non-abelian groups in general which have this property. In the latter case it has been noted that this number is three-fourths of the order of the group, but there is obviously no abelian group in which exactly three-fourths of the operators correspond to their inverses, while in the former case this number is p^{m-1} and every abelian group of order p^m has evidently also the property that exactly p^{m-1} of its operators correspond to their inverses in a possible automorphism of the group, since all the operators of every one of its subgroups of index p may correspond to their inverses when no other operator enjoys such a correspondence in the same automorphism. All these automorphisms are of order $2p$ and their number for each subgroup of index p is $p^\alpha - 1$, where α represents the number of the invariants of the subgroup. Hence the total number of such automorphisms for a given abelian group of order p^m is the sum of the numbers $p^\alpha - 1$ when α represents successively the number of the invariants of all the subgroups of index p.

[1] G. A. Miller, these PROCEEDINGS, **15**, 369, 1929.